ON THE POINT-LINE AS ELEMENT OF SPACE: A STUDY OF THE CORRESPONDING BILINEAR CONNEX*

BY

EDWARD KASNER

In the last paper published during his lifetime, CLEBSCH† enriched the analytical geometry of the plane by the introduction of a new form, the connex, which includes as very special cases both the curve considered as point locus and the curve considered as line envelope. The connex (m, n), of the m-th order and n-th class, is represented by an equation $f(x_1, x_2, x_3; u_1, u_2, u_3) = 0$, involving a set of point coördinates and a set of line coördinates. It may be defined as a manifold of ∞^3 elements each consisting of a point and a line. CLEBSCH studied the case $(1, 1)$, equivalent to a collineation; and with his pupil GODT, the case $(1, n)$.‡ The general connex (m, n) has been only incompletely investigated, principally in connection with the theory of algebraic differential equations.§

An extension to space was first proposed by KRAUSE,‖ who took for element the combination of a point and plane. An equation

$$f(x_1, x_2, x_3, x_4; u_1, u_2, u_3, u_4) = 0$$

represents a manifold of ∞^5 elements which may be termed for distinction a *point-plane connex*. KRAUSE confined himself to the case (2, 1). The general connex (m, n) has been studied, in an elaborate memoir, by SINTSOF.¶

* Presented to the Society at the Evanston meeting, September 2, 1902. Received for publication April 11, 1903.

† *Ueber ein neues Grundgebilde der analytischen Geometrie der Ebene*, Göttinger Nachrichten, 1872; reprinted, Mathematische Annalen, vol. 6 (1873), pp. 203–225.

‡ Cf. CLEBSCH-LINDEMANN, *Vorlesungen über Geometrie*, vol. 1 (1876), 2d section; GODT, *Ueber den Connex erster Ordnung und zweiter Classe*, Göttingen, 1873.

§ DARBOUX, Bulletin des Sciences Mathematiques, ser. 2, vol. 2 (1878); STEPHANOS, *ibid.*, vol. 4 (1880).

‖ *Ueber ein Gebilde der analytischen Geometrie des Raumes, welche dem Connexe zweiter Ordnung und erster Classe entspricht*, Mathematische Annalen, vol. 14 (1879), pp. 294–322.

¶ *Theory of the connex in space, in connection with the theory of partial differential equations of the first order* (Russian), Publications of Kasan University, 1895, 254 pp. This is known to the writer only through the author's summary in the Fortschritte der Mathematik. Cf. *Theorie des connexes dans l'espace*, Bulletin des Sciences Mathematiques, 1898, p. 176.

A different extension to space is made in the present paper. We have namely, as the simple elements of space, not only the point and the plane, but also the line; for compound element we may take then not only the point-plane, but also the point-line or the plane-line.* The last combination however is merely the dual of the second, and therefore requires no separate study. The combination of a point and a line, represented analytically by the coördinates $(x_1, x_2, x_3, x_4;\ p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$, will be termed simply an *element*. The total number of elements in space is ∞^7, so that in the corresponding geometry space is seven dimensional. An equation

$$f(x; p) \equiv a_x^m A_p^m = 0,$$

represents then a manifold of ∞^6 elements, which will be termed a *point-line connex of space*, or, when there is no ambiguity, simply a connex.

If we regard the point x as fixed, the equation above is satisfied by ∞^3 lines, namely those lines which combined with the given point constitute elements of the connex; these lines form a complex (of nth order) which will be referred to as *the complex corresponding to the point x*. On the other hand, if the line p is regarded as fixed, there are ∞^2 points which may be combined with it so as to give elements of the connex; these points form a surface (of mth order) which we term the *surface corresponding to the line p*.

After the study of manifolds defined by a single equation $f(x; p) = 0$, we might consider the manifolds of smaller dimension defined by two or more equations. Thus two equations $f_{m,n} = 0$, $f'_{m',n'} = 0$ represent ∞^5 elements, namely, those common to the corresponding connexes $(m, n), (m', n')$. To each point there corresponds, in this manifold, a congruence of lines, and to each line, a curve. The order of the congruence is nn' and the order of the curve is mm'.

Three equations $f_{m,n} = 0$, $f'_{m',n'} = 0$, $f''_{m'',n''} = 0$ represent a manifold of ∞^4 elements. The lines corresponding to a given point constitute a regulus of order $2nn'n''$; and the points corresponding to a given line form a discrete set, the number being $mm'm''$. Four equations represent ∞^3 elements, etc.

Returning to the case of a single equation $f = 0$ representing a connex, we proceed to define what may be termed, in analogy with CLEBSCH's terminology, its *principal coincidence*. This is composed of these elements of the connex for which the point x and the line p are mutually incident. Incidence of a point and line is a two-fold condition, whose complete expression requires, however, four equations† $\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = 0$. The principal coincidence consists then of the ∞^4 elements satisfying the equations

$$f = 0, \qquad \Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = 0.$$

---

*To complete the series, another compound element must be considered, namely the combination of a point, a line, and a plane. This the author expects to do elsewhere.

†See § 4, equations (13).

To each point there corresponds a single infinity of lines passing through it, forming a conical surface. This suggests an intimate relation between the geometry of point-line connexes and the theory of total or MONGE * differential equations in three variables, which the author expects to discuss elsewhere.

The object of the present paper is to study the simplest and most important case of the new type of connex, namely the case (1, 1). This is defined by a bilinear quaternary-senary form equated to zero. The complex corresponding to a point is now linear, and the surface corresponding to a line reduces to a plane. We have thus two linear transformations, one from a point to a linear complex, and the other from a line to a plane. These are discussed in §§ 1–3 from the point of view of distinct spaces, and in §§ 3–6 from that of superposed spaces, the last section being devoted to the principal coincidence. In the next section a covariant point-plane connex of type (2, 1) is derived. Normal forms with respect to both digredient and cogredient transformations are deduced in § 9 and § 11.

The basis of a purely synthetic treatment is furnished in § 8, where it is shown that the bilinear connex is equivalent to a correspondence between the points of a quadric surface and the lines of a linear congruence. This suggests the isomorphism of certain well-known projective groups, and the equivalence of the corresponding geometries (§ 10). In § 11 it is shown that the connex is completely determined by its fundamental configuration of five arbitrary points and two arbitrary lines. The absolute invariants of the connex are obtained in § 12.

§ 1. *The related linear transformations* F', F''.

The connex to be studied is defined by an equation of the type †

$$F \equiv \Sigma a_{i,kl} x_i P_{kl} = 0, \tag{1}$$

involving linearly the four point coördinates x_i and the six line coördinates P_{kl}. Symbolically,

$$F \equiv a_x A_P,$$

where

$$a_x \equiv a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4,$$

$$A_P \equiv A_{12} P_{12} + A_{13} P_{13} + A_{14} P_{14} + A_{34} P_{34} + A_{42} P_{42} + A_{23} P_{23},$$

so that

$$a_i A_{kl} = a_{i,kl} = - a_i A_{lk} = - a_{i,lk}.$$

* Cf. LIE-SCHEFFERS, *Geometrie der Berührungstransformationen*, Leipzig, 1896, chap. 7.

† Throughout the paper the indices i, k, l are understood to take the values 1, 2, 3, 4, and the summations extend over all the indices involved.

The ∞^2 points, each of which, combined with a fixed line P, gives an element of the connex, constitute a plane

$$u_i = a_i A_P. \tag{2}$$

This will be termed the plane corresponding to the line P; and the linear transformation (2), *which defines the passage from P to u, will be called the transformation F'.*

Consider next, in connection with a fixed point x, the lines which may be combined with it so as to give elements of the connex. There are ∞^3 such lines constituting a linear complex $\Sigma L_{ik} P_{ik} = 0$, the coefficients being

$$L_{ik} = A_{ik} a_x. \tag{3}$$

This is the linear complex corresponding to the point x; the equations (3) *define the linear transformation from x to L, which we shall refer to as the transformation F''.*

We then have, in connection with the general bilinear connex F, two linear transformations F' and F'', each of which determines the other and also F. *The study of F may therefore be regarded as equivalent to that of either of the transformations* (2) *or* (3).

§2. *The transformation F' and the two fundamental lines.*

The transformation (2), written out in extenso, is

$$\begin{aligned}
u_1 &= a_{1,12}P_{12} + a_{1,13}P_{13} + a_{1,14}P_{14} + a_{1,34}P_{34} + a_{1,42}P_{42} + a_{1,23}P_{23},\\
u_2 &= a_{2,12}P_{12} + a_{2,13}P_{13} + a_{2,14}P_{14} + a_{2,34}P_{34} + a_{2,42}P_{42} + a_{2,23}P_{23},\\
u_3 &= a_{3,12}P_{12} + a_{3,13}P_{13} + a_{3,14}P_{14} + a_{3,34}P_{34} + a_{3,42}P_{42} + a_{3,23}P_{23},\\
u_4 &= a_{4,12}P_{12} + a_{4,13}P_{13} + a_{4,14}P_{14} + a_{4,34}P_{34} + a_{4,42}P_{42} + a_{4,23}P_{23}.
\end{aligned} \tag{4}$$

The plane u, into which any line P is transformed, is unique unless all its coördinates vanish, i. e., unless

$$a_1 A_P = a_2 A_P = a_3 A_P = a_4 A_P = 0. \tag{5}$$

These linear equations define two lines, say Π', Π'', so that we may write

$$a_i A_{\Pi'} = 0, \quad a_i A_{\Pi''} = 0. \tag{6}$$

The transformation F' thus possesses two fundamental lines Π', Π'' having the property that the corresponding plane is indeterminate.

In connection with the original connex F the property of these lines may be restated: Each of the fundamental lines combined with an arbitrary point forms an element belonging to F.

The correspondence between the lines P and the planes u, defined by (4), is not uniquely reversible, as is evident from the fact that there are ∞^4 lines, but only ∞^3 planes. To each plane there corresponds indeed an infinity of lines whose locus we proceed to consider. The lines corresponding to u satisfy the equations

$$\frac{a_1 A_P}{u_1} = \frac{a_2 A_P}{u_2} = \frac{a_3 A_P}{u_3} = \frac{a_4 A_P}{u_4}. \tag{7}$$

These equations define a regulus * (half-quadric) which, in virtue of (6), contains the lines Π' and Π''.

Conversely, consider the regulus determined by a given line P and the fundamental lines. The coördinates of any line of this regulus are of the form

$$\lambda P_{ik} + \lambda' \Pi_{ik} + \lambda'' \Pi''_{ik}.$$

The corresponding plane has the coördinates

$$a_i A_{\lambda P + \lambda' \Pi' + \lambda'' \Pi''} = \lambda a_i A_P + \lambda' a_i A_{\Pi'} + \lambda'' a_i A_{\Pi''},$$

which by (6) reduce simply to $\lambda a_i A_P$, namely the coördinates of the plane corresponding to P. The result obtained may be stated as follows:

The lines of any regulus which contains the fundamental lines Π', Π'' *all transform by* F' *into the same plane. Conversely, corresponding to any plane* u, *there are an infinite number of lines, constituting a regulus passing through* Π', Π''.

§ 3. *The transformation* F'' *and the quadric* ω.

Written out in extenso, the second transformation, defined by (3), becomes

$$\begin{aligned}
L_{12} &= a_{1,12} x_1 + a_{2,12} x_2 + a_{3,12} x_3 + a_{4,12} x_4, \\
L_{13} &= a_{1,13} x_1 + a_{2,13} x_2 + a_{3,13} x_3 + a_{4,13} x_4, \\
L_{14} &= a_{1,14} x_1 + a_{2,14} x_2 + a_{3,14} x_3 + a_{4,14} x_4, \\
L_{34} &= a_{1,34} x_1 + a_{2,34} x_2 + a_{3,34} x_3 + a_{4,34} x_4, \\
L_{42} &= a_{1,42} x_1 + a_{2,42} x_2 + a_{3,42} x_3 + a_{4,42} x_4, \\
L_{23} &= a_{1,23} x_1 + a_{2,23} x_2 + a_{3,23} x_3 + a_{4,23} x_4.
\end{aligned} \tag{8}$$

Here the quantities L are the coördinates of the linear complex corresponding to the point x, so that the complex itself is

$$L_P \equiv \Sigma L_{ik} P_{ik} = 0. \tag{9}$$

* The term *regulus* hereafter will be used in a special sense, denoting the system defined by three linear equations, that is, one set of generators of a quadric surface. Cf. the French term *demi-quadrique*.

The complex will be special only if its invariant

$$[L, L] \equiv 2(L_{12}L_{34} + L_{13}L_{42} + L_{14}L_{23}) \tag{10}$$

vanishes. Substituting the values of L, the condition takes the form

$$\omega \equiv \omega_x^2 \equiv [A, A']a_x a'_x = 0, \tag{11}$$

where

$$[AA'] = A_{12}A'_{34} + A_{13}A'_{42} + A_{14}A'_{23} + A_{34}A'_{12} + A_{42}A'_{13} + A_{23}A'_{14}.$$

Therefore,

The points which are transformed into special linear complexes form a quadric surface $\omega = 0$.

The transformation F'' produces, from all the points of space, ∞^3 linear complexes, which are now to be characterized. The six linear functions $A_{ik}a_x$ of the four variables x_i must be connected by two linear relations. These are in fact

$$L_{\Pi'} = 0, \qquad L_{\Pi''} = 0.$$

For if we substitute the values of L in terms of x, we have for example $L_{\Pi'} = a_x A_{\Pi'}$, which vanishes by (6).

The totality of linear complexes L, *corresponding to all the points of space, constitute the linear three parameter system apolar* * *to* Π', Π''.

This system includes ∞^2 special complexes, whose directrices are of course the tractors of the fundamental lines Π', Π''. These correspond to the points of the quadric ω. †

§ 4 *The five fundamental points* O_x.

We have seen that to each point x, of the quadric ω, there corresponds by F' a special linear complex; which may be regarded as equivalent to simply a line (its directrix). The coördinates of this line are

$$P_{12} = A_{34}a_x, \; P_{13} = A_{42}a_x, \text{ etc.} \tag{12}$$

We inquire now concerning the points for which the corresponding line is an incident line, i. e., passes through the original point.

The general conditions expressing the incidence of a point x_i and a line p_{ik} are

$$\begin{aligned} \Omega_1 &\equiv x_2 p_{34} + x_3 p_{42} + x_4 p_{23} = 0, \\ -\Omega_2 &\equiv x_3 p_{41} + x_4 p_{13} + x_1 p_{34} = 0, \\ \Omega_3 &\equiv x_4 p_{12} + x_1 p_{24} + x_2 p_{41} = 0, \\ -\Omega_4 &\equiv x_1 p_{23} + x_2 p_{31} + x_3 p_{12} = 0. \end{aligned} \tag{13}$$

* In this connection the apolar relation signifies that the lines Π', Π'' are common to all the linear complexes L.

† For the detailed study of this correspondence see § 8 below.

The bilinear forms Ω_i are connected by the relation

$$x_1\Omega_1 + x_2\Omega_2 + x_3\Omega_3 + x_4\Omega_4 \equiv 0. \tag{13'}$$

Applying these conditions to the case of a point x_i (of the surface ω) and its corresponding line P_{ik} as given by (12), we obtain

$$\begin{aligned} q_1 &\equiv (x_2A_{12} + x_3A_{13} + x_4A_{14})a_x = 0, \\ q_2 &\equiv (x_3A_{23} + x_4A_{24} + x_1A_{21})a_x = 0, \\ q_3 &\equiv (x_4A_{34} + x_1A_{31} + x_2A_{32})a_x = 0, \\ q_4 &\equiv (x_1A_{41} + x_2A_{42} + x_3A_{43})a_x = 0. \end{aligned} \tag{14}$$

The quadrics q_i are connected by the relation

$$x_1q_1 + x_2q_2 + x_3q_3 + x_4q_4 \equiv 0. \tag{14'}$$

The question now resolves itself into the discussion of the points common to the five quadrics

$$\omega = 0, \qquad q_1 = 0, \qquad q_2 = 0, \qquad q_3 = 0, \qquad q_4 = 0.$$

In the first place, the points common to the last four necessarily lie on the first. For by introducing the values $L_{ik} = A_{ik}a_x$, q_1 for example may be written

$$L_{12}x_2 + L_{13}x_3 + L_{14}x_4 = 0.$$

As a consequence of the four equations $q_i = 0$, we then have

$$\begin{vmatrix} 0 & L_{12} & L_{13} & L_{14} \\ L_{21} & 0 & L_{23} & L_{24} \\ L_{31} & L_{32} & 0 & L_{34} \\ L_{41} & L_{42} & L_{43} & 0 \end{vmatrix} = 0,$$

which reduces to

$$L_{12}L_{34} + L_{13}L_{42} + L_{14}L_{23} = 0;$$

this, however, in virtue of the values of L_{ik}, is simply another form of (11), the equation of the quadric $\omega = 0$.

It remains to consider the points common to the four quadrics $q_i = 0$. The quadrics $q_1 = 0$, $q_2 = 0$, $q_3 = 0$ intersect in 8 points. Any point common to these three, by (14'), will also lie on $q_4 = 0$, unless $x_4 = 0$. We must exclude then those among the 8 points which lie in the plane $x_4 = 0$. This plane cuts the first three quadrics in the conics

$$\begin{aligned} \bar{q}_1 &\equiv (x_2A_{12} + x_3A_{13})\bar{a}_x = 0 \\ \bar{q}_2 &\equiv (x_3A_{23} + x_1A_{21})\bar{a}_x = 0 \\ \bar{q}_3 &\equiv (x_1A_{31} + x_2A_{32})\bar{a}_x = 0 \end{aligned} \qquad (\bar{a}_x = a_1x_1 + a_2x_2 + a_3x_3),$$

where the ternary forms are connected by

$$x_1\bar{q} + x_2\bar{q}_2 + x_3 q_3 = 0.$$

It follows that these conics have 3 points in common; for of the 4 points in which the first two intersect, that point must be excluded for which $x_3 = 0$. We have then finally $8 - 3 = 5$ points common to the four quadrics $q_i = 0$.

There are five points O_1, O_2, O_3, O_4, O_5 each of which (O_κ) has the property that the linear complex which corresponds by F''' is special and has for directrix an incident line (N_κ). These points lie on the quadric $\omega = 0$.

§5. *The congruence* Γ *and the surface* σ.

We return now to the transformation F' and inquire concerning the lines P for which the corresponding plane u is incident. The general conditions of incidence are

$$\begin{aligned} u_2 P_{12} + u_3 P_{13} + u_4 P_{14} &= 0, \\ u_3 P_{23} + u_4 P_{24} + u_1 P_{21} &= 0, \\ u_4 P_{34} + u_1 P_{31} + u_2 P_{32} &= 0, \\ u_1 P_{41} + u_2 P_{42} + u_3 P_{43} &= 0. \end{aligned} \tag{15}$$

Substituting then $u_i = a_i A_P$, we obtain, as conditions on the lines P in question,

$$\begin{aligned} \Gamma_1 &\equiv (a_2 P_{12} + a_3 P_{13} + a_4 P_{14}) A_P = 0, \\ \Gamma_2 &\equiv (a_3 P_{23} + a_4 P_{24} + a_1 P_{21}) A_P = 0, \\ \Gamma_3 &\equiv (a_4 P_{34} + a_1 P_{31} + a_2 P_{32}) A_P = 0, \\ \Gamma_4 &\equiv (a_1 P_{41} + a_2 P_{42} + a_3 P_{43}) A_P = 0. \end{aligned} \tag{16}$$

The lines P, characterized by the property that the plane which corresponds by F' is incident, form a congruence Γ defined by the equations (16).

What is the order and the class of this congruence? Consider any point, for convenience say $(1, 0, 0, 0)$. If the line P of the congruence passes through this point, then in the first place $P_{34} = P_{42} = P_{23} = 0$; and in the second place the remaining three coördinates satisfy the equations

$$\begin{aligned} (a_2 P_{12} + a_3 P_{13} + a_4 P_{14})(A_{12} P_{12} + A_{13} P_{13} + A_{14} P_{14}) &= 0, \\ a_1 P_{12}(A_{12} P_{12} + A_{13} P_{13} + A_{14} P_{14}) &= 0, \\ a_1 P_{13}(A_{12} P_{12} + A_{13} P_{13} + A_{14} P_{14}) &= 0, \\ a_1 P_{14}(A_{12} P_{12} + A_{13} P_{13} + A_{14} P_{14}) &= 0, \end{aligned} \tag{17}$$

obtained from (16). From the last three equations of this set, since P_{12}, P_{13}, P_{14} cannot all vanish (for then all six coördinates would vanish), it follows that

$$a_1(A_{12}P_{12} + A_{13}P_{13} + A_{14}P_{14}) = 0,$$

which, in combination with the first equation of (17), shows that there are two solutions. Since the coördinate system is entirely arbitrary, the point $(1, 0, 0, 0)$ is really a point in general position, and it has thus been proved that through the general point there pass two lines of the congruence. The order of Γ is therefore equal to two.

The class of the congruence is determined by finding the number of lines contained in a general plane, say $x_1 = 0$. We have now $P_{12} = P_{13} = P_{14} = 0$; the first equation of (16) is then identically satisfied, and the other equations, yield three quadratic equations in P_{32}, P_{42}, P_{23}, whose discussion shows that there are three common solutions. The class of Γ is therefore equal to three.

The congruence Γ is of second order and third class. It follows that the number of congruence lines cutting a given line is equal to five.

To the double infinity of lines Γ there corresponds, by F', a double infinity of planes which is now to be investigated. For this purpose, consider the equations (15) in connection with the system $\rho u_i = a_i A_P$, and eliminate the constant factor ρ and the coördinates P_{ik}. The result may be written

$$(18)\qquad \begin{vmatrix} a_{1,12} & a_{1,13} & a_{1,14} & a_{2,34} & a_{1,42} & a_{1,23} & u_1 \\ a_{2,12} & \cdot & \cdot & \cdot & \cdot & \cdot & u_2 \\ a_{3,12} & \cdot & \cdot & \cdot & \cdot & \cdot & u_3 \\ a_{4,12} & \cdot & \cdot & \cdot & \cdot & \cdot & u_4 \\ u_2 & u_3 & u_4 & 0 & 0 & 0 & 0 \\ -u_1 & 0 & 0 & 0 & -u_4 & u_3 & 0 \\ 0 & -u_1 & 0 & u_4 & 0 & -u_2 & 0 \\ 0 & 0 & -u_1 & -u_3 & u_2 & 0 & 0 \end{vmatrix} = 0.$$

If we denote the determinants of seventh order, obtained by leaving out in turn the 5th, 6th, 7th and 8th row, by σ_1, σ_2, σ_3 and σ_4 respectively, it may be shown that these determinants are factorable. For example σ_4 vanishes if $u_4 = 0$, and therefore contains u_4 as a factor. The result of the detailed discussion is that there is a factor σ, of the third degree in u, which is common to all four determinants σ_i; so that

$$(19)\qquad \sigma \equiv \frac{\sigma_1}{u_1} \equiv \frac{\sigma_2}{u_2} \equiv \frac{\sigma_3}{u_3} \equiv \frac{\sigma_4}{u_4}.$$

The planes which correspond to the lines of the congruence Γ envelope a surface of third class $\sigma = 0$, where σ is defined by (19).

§ 6. *The principal coincidence and the covariant K.*

The principal coincidence of the connex F consists, by definition, of the ∞^4 incident elements belonging to F, i. e., of those elements in which the point and the lines are incident. Connected with each point x there is a single infinity of lines P; these are the lines of the corresponding linear complex L, defined in § 3, which pass through x. These lines lie in a plane U, the polar plane of x in the null system determined by L. The coördinates of U are given by

$$
\begin{aligned}
U_1 &= L_{12}x_2 + L_{13}x_3 + L_{14}x_4 = (A_{12}x_2 + A_{13}x_3 + A_{14}x_4)a_x, \\
U_2 &= L_{23}x_3 + L_{24}x_4 + L_{21}x_1 = (A_{23}x_3 + A_{24}x_4 + A_{21}x_1)a_x, \\
U_3 &= L_{34}x_4 + L_{31}x_1 + L_{32}x_2 = (A_{34}x_4 + A_{31}x_1 + A_{32}x_2)a_x, \\
U_4 &= L_{41}x_1 + L_{42}x_2 + L_{43}x_3 = (A_{41}x_1 + A_{42}x_2 + A_{43}x_3)a_x.
\end{aligned} \tag{20}
$$

Comparing with (14), these may be abreviated

$$U_i = q_i = q_{ix}^2. \tag{20'}$$

The principal coincidence is completely defined by this quadratic transformation from the point x to the plane U. The transformation may also be expressed by means of the vanishing of a form involving two sets of point coördinates, namely by

$$
\begin{aligned}
K &\equiv k_x^2 K_X \equiv \Sigma q_{ix}^2 X_i \\
&\equiv \left\{ \begin{array}{l} (A_{12}x_2 + A_{13}x_3 + A_{14}x_4)X_1 + (A_{23}x_3 + A_{24}x_4 + A_{21}x_1)X_2 \\ + (A_{34}x_4 + A_{31}x_1 + A_{32}x_2)X_3 + (A_{41}x_1 + A_{42}x_2 + A_{43}x_3)X_3 \end{array} \right\} a_x = 0.
\end{aligned} \tag{21}
$$

The vanishing of the covariant K may therefore be interpreted: to each point x there corresponds a plane U (passing through it) containing the pencil of lines which, combined with x, constitute elements of the principal coincidence. The plane U is the polar of x with respect to the linear complex L corresponding to x by F'''.

The plane U ceases to be determined when the complex L is special and has its directrix passing through x. This we have seen occurs only for the five points O_1, O_2, O_3, O_4, O_5 defined in § 3. Therefore *the principal coincidence of F has five fundamental points $O_1, \cdots, O_5$; with each of these may be combined not merely a pencil of lines, but the entire bundle of lines through the point.*

We consider now the inverse of the transformation defined by (20) or (20′), and prove that *there are in general three points x which correspond to the same plane U.* The points x are in fact determined by the equations

$$\frac{q_1}{U_1} = \frac{q_2}{U_2} = \frac{q_3}{U_2} = \frac{q_4}{U_4},$$

or by the equivalent set

$$U_1 q_2 - U_2 q_1 = 0, \quad U_1 q_3 - U_3 q_1 = 0, \quad U_1 q_4 - U_4 q_3 = 0.$$

These quadratic equations in x, for all values of U_i, are satisfied by the coördinates of the five points O_κ, since the latter cause all the quadrics q_i to vanish. There are then only three variable solutions of the system—which proves the result stated above.

By means of $K = 0$, to each point X there corresponds a quadric surface

$$K_X k_x^2 \equiv X_1 q_1 + X_2 q_2 + X_3 q_3 + X_4 q_4 = 0$$

belonging to the linear system determined by q_1, q_2, q_3, q_4. *This quadric is the locus of the point x for which the corresponding plane U (by F') passes through the original point X.*

The locus of the point x whose corresponding plane U is incident is the cubic surface $k_x^2 K_x = 0$. The locus of the point X whose corresponding quadric surface is degenerate is the quartic surface

$$(k k' k'' k''')^2 K_X K'_X K''_X K'''_X = 0.$$

§ 7. *The covariant point-plane connex C.*

Consider the bundle of lines through any point X; for each of these lines we have from (13)

$$X_2 P_{34} + X_3 P_{42} + X_4 P_{23} = 0, \text{ etc.} \tag{22}$$

To each line of the bundle there corresponds, by F', a plane u, so that

$$a_i A_p - \rho u_i = 0. \tag{23}$$

We prove now that the double infinity of planes so obtained themselves constitute a bundle.

Eliminate ρ and the coördinates P_{ik} from the four equations (23) and any three of the set (22). We obtain thus four relations, expressed by the vanishing of the determinants in the matrix

$$\begin{vmatrix} a_{1,12} & a_{1,13} & a_{1,14} & a_{1,34} & a_{1,42} & a_{1,23} & u_1 \\ a_{2,12} & \cdot & \cdot & \cdot & \cdot & \cdot & u_2 \\ a_{3,12} & \cdot & \cdot & \cdot & \cdot & \cdot & u_3 \\ a_{4,12} & \cdot & \cdot & \cdot & \cdot & \cdot & u_4 \\ 0 & 0 & 0 & X_2 & X_3 & X_4 & 0 \\ 0 & -X_4 & X_3 & -X_1 & 0 & 0 & 0 \\ X_4 & 0 & -X_2 & 0 & -X_1 & 0 & 0 \\ -X_3 & X_2 & 0 & 0 & 0 & -X_1 & 0 \end{vmatrix} \tag{24}$$

which result from omitting in turn the fifth, sixth, seventh and eighth row. These determinants have redundant factors X_1, X_2, X_3, X_4 respectively; the factor common to all is of the second degree in X and of the first degree in u. The covariant thus obtained will be written

$$C \equiv C_X^2 u_c. \tag{25}$$

We thus obtain a covariant point-plane connex $C = 0$ of the second order and first class. In this connex, to each point X there corresponds a bundle of planes, which coincides with the bundle obtained by F' from the bundle of lines through X.

The coördinates of the vertex x of this bundle are given by

$$x_i = c_i C_X^2. \tag{26}$$

If the point x and the plane u together form an element of the connex $C = 0$, it follows from the above theorem that there exists a line through X whose corresponding plane (by F') is u; or, what is equivalent, that of the ∞ lines forming a regulus corresponding by F' to u, one passes through X.

The quadric surface which corresponds, in the connex $C = 0$, to a given plane u, contains the regulus of lines which are transformed by F' into the same plane u.

The general point-plane connex of second order and first class has been investigated by KRAUSE.* The results may be applied to the connex $C = 0$, and thus indirectly to the original point-line connex F. For example, the locus of the plane u for which the corresponding quadric is degenerate, is the surface of fourth class

$$(CC'C''C''')^2 u_c u_{c'} u_{c''} u_{c'''} = 0;$$

the left member is therefore a covariant of F, involving the coefficients of F to the sixteenth degree. The condition that the plane should be tangent to the corresponding quadric yields a surface of fifth class

$$(CC'C''u)^2 u_c u_{c'} u_{c''} = 0.$$

There are 15 points X each of which coincides with its corresponding point x.

The covariant C was obtained by considering the ∞^2 lines passing through a point X. In an entirely analogous manner, by considering the ∞^2 lines lying in a plane U, we obtain another covariant, involving two sets of plane coördinates, which we write

$$E \equiv U_E^2 u_c. \tag{27}$$

By means of $E = 0$, to a plane U there corresponds a point x such that the lines of U are transformed, by F', into the planes through x; again to a plane u

*See reference on page 213 above.

there corresponds a (class) quadric, containing the regulus of lines which F' transforms into u. It follows that the class quadric thus obtained is identical with the order quadric corresponding by $C = 0$ to the same plane u.

§8. *A geometrical construction (or definition) of the connex.**

If we interpret the x_i as point coördinates in a space s, and the P_{ik} as line coördinates in a distinct or superposed space S, then, connected with the general connex F, there is in s a quadric surface ω, and in S a pair of lines Π', Π'', such that to each point of ω there corresponds a tractor of Π', Π''. The transformation from the point x to the line P, as seen from (12), is linear, hence to a linear system of points corresponds a linear system of lines. But the only linear systems of points on ω are the generators; therefore to the points of a generator correspond lines forming a pencil. Since all the lines must be tractors of Π', Π'' it follows that the center of the pencil is a point on Π' or Π'' and that the plane of the pencil passes through Π'' or Π'. Generators of the same system do not intersect; hence the corresponding pencils have centers on the same line Π' or Π''. Thus there are established $(1, 1)$ correspondences between the generators of the first system on ω and the points of say Π'; and between the generators of the second system and the points of Π''. The tractor corresponding to any point of ω is of course the line joining the points on Π' and Π'' which correspond respectively to the two generators passing through the given point.

If now we take arbitrarily, in the space s, a quadric ω, and in the space S, a pair of lines Π', Π'', and establish $(1, 1)$ correspondences between the two systems of generators and the points of Π', Π''—or what is the same, establish a $(1, 1)$ correspondence between the points of ω and the tractors of Π', Π'', we employ $9 + 4 + 4 + 6 = 23$ parameters. This is the number of parameters involved in the general connex F. It is thus rendered plausible that the connex is defined by the figure described.

The connex is in fact obtained most simply as follows, in terms of the related transformation F'. Consider any line P in the space S. The tractors common to P, Π', Π'' establish a homographic correspondence between the points of Π' and Π''; the corresponding generators must also be in $(1, 1)$ correspondence; hence the locus of the points on ω corresponding to the tractors considered is a conic. Thus to the line P in S there corresponds finally a unique plane in s, namely the plane of the conic just obtained.

The transformation F''' is derived as follows. Consider any point x in s, with the bundle of planes passing through it. Each of these planes intersects ω in a conic; this conic establishes a $(1, 1)$ correspondence between the two systems of generators; the ranges on Π' and Π'' are thus placed in homographic

* This section furnishes the basis for a purely synthetic treatment of the bilinear connex.

correspondence; hence the joins of corresponding points constitute a regulus (half-quadric). The ∞^2 planes of the bundle yield in this way ∞^2 reguli, and the ∞^3 lines of these reguli constitute a linear complex. This is the linear complex L which corresponds to the point x by means of the transformation F''

Given a plane u, it was seen in § 2 that there is a regulus of lines which are all transformed by F'' into the same plane u. This regulus is now obtained as the conjugate regulus of that described in the preceding paragraph as derived from the conic in which u cuts ω.*

The connex F itself is composed of ∞^6 elements (x, P) each consisting of a point in s and a line in S; *these elements are characterized as follows: to the tractors of* P, Π', Π'' *correspond points on* ω *whose locus is a conic of which the plane passes through* x.

The results of this section may be stated:

The study of the general bilinear connex is equivalent to the study of the configuration composed of a general quadric surface and a linear congruence (tractors of a pair of lines), the points of the surface and the lines of the congruence being placed in $(1, 1)$ *correspondence.*

§ 9. *Normal form with respect to digredient transformation.*

From the preceding geometrical determination we may obtain a useful normal form. For this purpose it is sufficient to take the quadric ω, the pair of lines Π', Π'', and the correspondence, in the simplest fashion. Let the quadric in s be

$$\omega \equiv x_2x_3 - x_1x_4 = 0; \tag{28}$$

its generators are

$$\frac{x_1}{x_2} = \frac{x_3}{x_4} = \lambda, \qquad \frac{x_1}{x_3} = \frac{x_2}{x_4} = \mu, \tag{28'}$$

where λ, μ are parameters. In S let the two fundamental lines be

$$\begin{aligned} \Pi' &: \quad 0, 0, 1, 0, 0, 0, \\ \Pi'' &: \quad 0, 0, 0, 0, 0, 1. \end{aligned} \tag{29}$$

The correspondence we define by assigning to the generator of the first system with parameter λ, the point $(\lambda, 0, 0, 1)$ on the line Π'; and similarly, to the generator of the second system with parameter μ, the point $(0, \mu, 1, 0)$ on the

* From this it is seen that the regulus degenerates when, and only when, the plane u is tangent to ω; the result of the degeneration being two pencils having their vertices on Π', Π'' respectively. The totality of pencils so obtained constitute the quadratic complex

$$(a\omega\omega'\omega'')(a\omega'\omega\omega'')A_PA'_P = 0,$$

which therefore degenerates into the two linear complexes composed of the lines cutting Π' or Π''.

line Π''. To the point (λ, μ) of ω there corresponds then the tractor joining the points $(\lambda, 0, 0, 1)$, $(0, \mu, 1, 0)$, namely the line

$$\lambda\mu,\ \lambda,\ 0,\ -1,\ \mu,\ 0. \tag{30}$$

Consider any line P_{ik} in S. If the tractor (30) is to intersect this line, then

$$P_{34}\lambda\mu + P_{42}\lambda + P_{13}\mu - P_{12} = 0;$$

or, from (28′),

$$P_{34}x_1 + P_{13}x_2 + P_{42}x_3 - P_{12}x_4 = 0.$$

This is in fact the plane corresponding to the line P, so that the transformation F' takes the form

$$u_1 = P_{34},\ u_2 = P_{13},\ u_3 = P_{42},\ u_4 = -P_{12}. \tag{31}$$

The matrix of this transformation is

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \tag{32}$$

which, read by columns, gives for the transformation F'''

$$L_{12} = -x_4,\ L_{13} = x_2,\ L_{14} = 0,\ L_{34} = x_1,\ L_{42} = x_3,\ L_{23} = 0. \tag{33}$$

Finally the normal form of the connex is

$$F \equiv x_1P_{34} + x_2P_{13} + x_3P_{42} - x_4P_{12} = 0. \tag{34}$$

In the reduction to this form, it is assumed that ω is not degenerate and that Π', Π'' are distinct and non-intersecting. Consider first the condition that ω shall be degenerate, namely the vanishing of its discriminant J. We have

$$\omega \equiv \omega_x^2 \equiv [A\,A']\,a_x a'_x,$$

so that

$$J = (\omega\omega'\omega''\omega''')^2$$

$$= \begin{vmatrix} [AA']a_1a'_1 & [AA']a_1a'_2 & [AA']a_1a'_3 & [AA']a_1a'_4 \\ [A''A''']a''_2a'''_1 & [A''A''']a''_2a'''_2 & [A''A''']a''_2a'''_3 & [A''A''']a''_2a'''_4 \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{vmatrix}, \tag{35}$$

which, by permutation of the equivalent symbols, may finally be reduced to

$$J = [A\,A'][A''A'''][A^{(4)}A^{(5)}][A^{(6)}A^{(7)}] \\ (aa''a^{(4)}a^{(6)})(a'a'''a^{(5)}a^{(7)}). \tag{35′}$$

Consider in the second place the condition that Π', Π'' shall coincide. Since these are obtained as the lines common to four linear complexes, namely (20), the required condition is expressed by the vanishing of the combinant of these four complexes. This combinant is

$$\begin{vmatrix} (11) & (12) & (13) & (14) \\ (21) & (22) & (23) & (24) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

where (11), for example, denotes the invariant of $a_1 A_P = 0$, and (12), the simultaneous invariant of $a_1 A_P = 0$, $a_2 A_P = 0$; so that

$$(11) = a_1 a_1' [AA'], \qquad (12) = a_1 a_2' [AA'], \text{ etc.}$$

Introducing these values, the combinant reduces to the invariant J defined above. Therefore

When the quadric ω degenerates, the fundamental lines Π', Π'' coincide, and vice versa. The condition for either case is expressed by the vanishing of the invariant of the eighth degree J defined by (35′).*

When J does not vanish the initial representations (28), (28′), (29) are justified, and therefore

The connex F, by digredient transformation, may be reduced to the normal form (34), *provided the invariant J does not vanish.*†

§ 10. *The relation of certain projective groups.*

The discussion in § 8 leads to the consideration of a (1, 1) correspondence between the points of quadric surface and the lines of a linear congruence. This correspondence is of interest in itself and suggests a connection between the related transformations which we shall now consider.

A proper quadric surface is transformed into itself by a well known six parameter group of collineations, composed of two continuous series g_6, h_6 distinguished by the fact that the transformations of the first kind turn each system of generators into itself, while those of the second kind interchange the two systems. On the other hand, a proper linear congruence, consisting of the tractors of two non-intersecting lines, is converted into itself by a seven parameter group of collineations (G_7, H_7), the transformations of the first kind leaving each of the fundamental lines invariant while those of the second kind interchange them.

* To the points of ω correspond the congruence composed of the tractors of Π', Π''; the theorem above shows that the quadric and the congruence degenerate simultaneously.

† Moreover the reduction may be effected in ∞^7 distinct ways. This follows from the last theorem of § 10.

This group possesses a self-conjugate subgroup I_1 composed of the single infinity of skew perspectivities* which leave invariant the individual points of both fundamental lines.

In virtue of the correspondence between the elements, each transformation of the group (G_7, H_7) induces a transformation of the points of ω which is equivalent to a collineation of the automorphic group (g_6, h_6). The transformations of I_1 leave invariant each line of the congruence, and hence the induced transformation is merely identity. It follows that any two collineations of (G_7, H_7) which may be obtained from each other by a transformation I_1 induce the same collineation of (g_6, h_6).

The group (g_6, h_6) of a proper quadric surface is isomorphic with the group (G_7, H_7) of a proper linear congruence. The isomorphism is meriedric; to each transformation of the first group corresponds an infinity of transformations of the second, all of which are obtained from any one by means of the self-conjugate group I_1.

If we regard as identical, in the congruence group, those collineations which have the same effect upon the lines of the congruence (though they have different effects upon the points of space), the group reduces to one of six parameters $(\overline{G}_6, \overline{H}_6)$,† and the relation above may be restated: *The groups (g_6, h_6) and $(\overline{G}_6, \overline{H}_6)$ are simply isomorphic.*

The nature of the isomorphism is seen most clearly by observing that each point of the quadric, as well as each line of the congruence, may be represented by a pair of parameters (λ, μ), and that then both groups take the form

$$\lambda' = \frac{a\lambda + b}{c\lambda + d}, \qquad \mu' = \frac{\alpha\mu + \beta}{\gamma\mu + \delta};$$

$$\lambda' = \frac{a\mu + b}{c\mu + d}, \qquad \mu' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}.$$

Another point of view is obtained by making use of the well known representation of lines by points in a space R_5. All the lines of space being represented by a manifold M_4^2, a linear congruence is represented by the intersection of this manifold with a 3-flat, i. e., by an ordinary quadric surface.

The correspondence of elements and the isomorphism of groups justify the statement: *Geometry on a quadric surface is equivalent to geometry in a linear congruence.* Both may be adequately represented by means of double binary forms involving digredient variables.‡

* Such a collineation is defined by the property that the join of corresponding points meets two fixed lines and is divided by them in a fixed anharmonic ratio.

† This is of course equivalent to the quotient group of (G_7, H_7) over I_1.

‡ Cf. E. KASNER, *The cogredient and digredient theories of multiple binary forms*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 86–102.

The detailed study of the correspondence between the two geometries is of some interest and presents little difficulty. Thus to a conic corresponds an hyperboloid passing through the two fundamental lines; two conics intersect in two points, hence the corresponding hyperboloids intersect in two lines—in addition to the fundamental lines; to a twisted quartic of the first species corresponds a quartic surface having each of the fundamental lines for a double line,* etc.

We return now to the connex F, and inquire concerning its automorphic transformations. The spaces s and S are here regarded as distinct, undergoing independent collineations, so that the number of constants involved in a transformation, made up of a pair of collineations, is thirty. The number of conditions imposed by the invariance of F is 23, so that the automorphic group involves 7 parameters. The question is equivalent to the automorphic transformation of the configuration composed of the quadric ω, the linear congruence defined by Π', Π'', and the correspondence between the two.

Consider any collineation T of the group (G_7, H_7). This in virtue of the correspondence induces in the space s a collineation t of the group (g_6, h_6). The pair t, T then constitute a transformation of the required type. Since one pair arises from each collineation of the group (G_7, H_7) it follows that

The digredient transformations for which the general bilinear connex is invariant constitute a seven parameter group isomorphic with the group (G_7, H_7) of the general linear congruence.

§ 11. *The fundamental configuration, and a normal form with respect to cogredient transformation.*

It has been seen in § 4 that the bilinear connex has five fundamental points O_κ, for each of which the corresponding linear complex degenerates into an incident line N_κ. The configuration O_κ, N_κ is not entirely arbitrary since the lines N_κ are all tractors of the fundamental lines Π', Π''. There is however no other restriction, as is rendered plausible by an enumeration of constants; in fact the figure composed of five arbitrary points and two arbitrary lines involves $15 + 8$ constants, which is the number of constants involved in the general connex. We now prove precisely that

The connex is, in general, completely determined by the five arbitrary points O_1, O_2, O_3, O_4, O_5 and the two arbitrary lines Π', Π''.

We suppose that the points and lines above are in general position: specifi-

* The geometries above are also equivalent to the inversion geometry of the plane, so that to a circle corresponds an hyperboloid, etc. Cf. the author's article, *The invariant theory of the inversion group: geometry on a quadric surface*, Transactions of the American Mathematical Society, vol. 1 (1900), pp. 430–498. The results as to cyclic curves in chapter V of this article may easily be translated into theorems regarding ruled surfaces of the fourth order.

cally, that the two lines do not intersect; that none of the points lies on either line; that no three of the points are collinear; and that none of the lines joining two of the points intersects both of the given lines. It follows from these assumptions that the lines N_κ, namely the tractors of Π', Π'' passing through the points O_κ, are all distinct and intersect Π' in distinct points A'_κ, and Π'' in distinct points A''_κ. We proceed now with the proof of the theorem stated.

In the first place, the quadric surface ω is completely determined as follows: It must pass through the five points O_κ, and in such a way that the generators g'_κ of the first system passing through these points are homographic to the points A'_κ, and that the generators g''_κ of the second system are homographic to the points A''_κ. This is in fact equivalent to nine linear conditions. The quadric ω may be constructed synthetically quite simply. Take any non-degenerate quadric $\bar{\omega}$, and upon it any three points $\bar{O}_1$, $\bar{O}_2$, $\bar{O}_3$, no two lying on the same generator; denote the generators through these points by $\bar{g}'_1, \bar{g}'_2, \bar{g}'_3$ and $\bar{g}''_1, \bar{g}''_2, \bar{g}''_3$. In the first* system find the two generators $\bar{g}'_4, \bar{g}'_5$ so that the five generators $\bar{g}'_\kappa$ shall be homographic to the points A'_κ; similarly find $\bar{g}''_4, \bar{g}''_5$. Denote the intersection of $\bar{g}'_4, \bar{g}''_4$ by $\bar{O}_4$, and that of $\bar{g}'_5, \bar{g}''_5$ by $\bar{O}_5$. Since the generators $\bar{g}'_4, \bar{g}''_4$ are all distinct it follows that the points $\bar{O}_\kappa$ are all distinct and that no three are collinear. The same is true by hypothesis of the given points O_κ. There must exist then a unique collineation which transforms the points $\bar{O}_\kappa$ in the points O_κ. This collineation converts $\bar{\omega}$ into the required quadric ω.

In the second place, there is a definite correspondence between the points of ω and the tractors of Π', Π''. For we have corresponding to g'_1, g'_2, g'_3 the points A'_1, A'_2, A'_3, and to g''_1, g''_2, g''_3, the points A''_1, A''_2, A''_3. This defines the correspondence; then in virtue of the above construction A'_4, A'_5 correspond to g'_4, g'_5, and A''_4, A''_5 correspond to g''_4, g''_5.

Having now determined the quadric ω, the congruence defined by Π', Π'', and the correspondence between the two, it follows from § 9 that the connex itself is uniquely determined, which completes the proof of the theorem.

It is of interest to carry out the argument on analytic lines as it leads to a normal form for F. Take for the coördinate tetrahedron that defined by the points O_1, O_2, O_3, O_4 and let O_5 be the unit point. The coördinates of the five fundamental points are thus

$$\begin{array}{ccccc} O_1 & O_2 & O_3 & O_4 & O_5 \\ (1,0,0,0) & (0,1,0,0) & (0,0,1,0) & (0,0,0,1) & (1,1,1,1). \end{array}$$

Let the coördinates of Π', Π'' be λ'_{ik}, λ''_{ik} respectively. The problem is now to find the connex F determined by these fundamental points and lines, i. e., to

* Which system is called first, is a matter of indifference; for the interchange of the two systems, it may be shown, does not affect the final result.

express the 24 coefficients $a_{i,kl}$ in terms of λ'_{ik}, λ''_{ik}. The transformation F' carries the point O_1 into the line whose coördinates are

$$a_{1,12}, \quad a_{1,13}, \quad a_{1,14}, \quad a_{1,34}, \quad a_{1,42}, \quad a_{1,23};$$

since this is to be a line through O_1, it follows from the conditions for incidence that the last three coördinates vanish. Dealing similarly with O_2, O_3, O_4, it is found that the matrix of F is of the form

$$(36) \qquad \begin{vmatrix} 0 & 0 & 0 & a_{1,34} & a_{1,42} & a_{1,23} \\ 0 & a_{2,13} & a_{2,14} & a_{2,34} & 0 & 0 \\ a_{3,12} & 0 & a_{3,14} & 0 & a_{3,42} & 0 \\ a_{4,12} & a_{4,13} & 0 & 0 & 0 & a_{4,23} \end{vmatrix}.$$

The general connex contains 23 constants; by a proper collineation it should be possible to reduce the number to $23 - 15 = 8$. The matrix above, involving 12 homogeneous constants, is thus equivalent to only a semi-normal form of the connex. A reduced normal form is obtained by expressing the twelve coefficients of the matrix in terms of λ'_{ik}, λ''_{ik}, which are in fact equivalent to only eight essential constants, since each sextuple of coördinates is connected by a quadratic relation. This reduction will now be carried out as follows.

The line into which O_1 is transformed, namely $(a_{1,34}, a_{1,42}, a_{1,23}, 0, 0, 0)$ is to be a tractor of Π', Π'', hence

$$\lambda'_{34}a_{1,34} + \lambda'_{42}a_{1,42} + \lambda'_{23}a_{1,23} = 0,$$
$$\lambda''_{34}a_{1,34} + \lambda''_{42}a_{1,42} + \lambda''_{23}a_{1,23} = 0.$$

Similarly from O_2, O_3, O_4 we find

$$\lambda'_{13}a_{2,13} + \lambda'_{14}a_{2,14} + \lambda'_{34}a_{2,34} = 0,$$
$$\lambda''_{13}a_{2,13} + \lambda''_{14}a_{2,14} + \lambda''_{34}a_{2,34} = 0;$$
$$\lambda'_{12}a_{3,12} + \lambda'_{14}a_{2,14} + \lambda'_{42}a_{3,42} = 0,$$
$$\lambda''_{12}a_{3,12} + \lambda''_{12}a_{3,14} + \lambda''_{42}a_{3,42} = 0;$$
$$\lambda'_{12}a_{4,12} + \lambda'_{13}a_{4,13} + \lambda'_{23}a_{4,23} = 0,$$
$$\lambda''_{12}a_{4,12} + \lambda''_{13}a_{4,13} + \lambda''_{23}a_{4,23} = 0.$$

Finally, since O_5 is to be transformed into an incident line, we have the additional conditions

$$a_{2,13} + a_{2,14} + a_{3,12} + a_{3,14} + a_{4,12} + a_{4,13} = 0,$$
$$a_{3,24} + a_{3,21} + a_{4,23} + a_{4,21} + a_{1,23} + a_{1,24} = 0,$$
$$a_{4,31} + a_{4,32} + a_{1,34} + a_{1,32} + a_{2,34} + a_{2,31} = 0,$$
$$a_{1,42} + a_{1,43} + a_{2,41} + a_{2,43} + a_{3,41} + a_{3,42} = 0,$$

of which, however, only three are independent. Taking then any three of this set with the eight equations above, we have eleven linear equations for the determination of the twelve homogeneous coefficients in terms of λ'_{ik}, λ''_{ik}. The substitution of these values in the matrix (36) gives the reduced normal form of the matrix, and thus the final normal form of the connex F.

§ 12 *The absolute invariants of the connex.*

To obtain the absolute invariants of F, we may employ its covariant figures, in particular the quadric ω, the five points O_κ, the lines Π', Π'', and the tractors N_κ. The line N_κ cuts Π' and Π'' in A'_κ and A''_κ; it intersects the quadric ω in the point O_κ and in an additional point say $\bar{O}_\kappa$. Through O_κ there pass two generators of ω, say g'_κ, g''_κ; and through $\bar{O}_\kappa$, two generators $\bar{g}'_\kappa$, $\bar{g}''_\kappa$.

The number of independent absolute invariants of F is $23 - 15 = 8$. These may be represented as anharmonic ratios by means of the covariant elements above. The five points A'_κ on Π' give two independent ratios, and so do the five points A''_κ on Π''. In virtue of the (1, 1) correspondences between the points of Π' and the generators of the system g', and between the points Π'' and the generators of the system g'', it follows that

$$(g'_1, g'_2, g'_3, g'_4, g'_5) \; \overline{\wedge} \; (A'_1, A'_2, A'_3, A'_4, A'_5),$$
$$(g''_1, g''_2, g''_3, g''_4, g''_5) \; \overline{\wedge} \; (A''_1, A''_2, A''_3, A''_4, A''_5),$$

where the symbol denotes the equality of corresponding anharmonic ratios. Additional absolute invariants are obtained by considering however the generators $\bar{g}'_\kappa$, $\bar{g}''_\kappa$, so that

A set of eight independent absolute invariants is obtained by taking a pair of anharmonic ratios from each of the four quintuples g'_κ, g''_κ, $\bar{g}'_\kappa$, $\bar{g}''_\kappa$. All the absolute invariants of the connex are expressible in terms of these eight.

Another set is obtained by considering the points in which ω intersects Π', Π''. On each line Π', Π'' we have then seven points (two arising from ω and five from the tractors N_κ). Seven collinear points, however, have four independent anharmonic ratios, so that we obtain in all the required eight.

Again, absolute invariants may be defined, though not so symmetrically, without making use of the quadric ω. This must be possible, since it was seen in § 8 that the fundamental figure O_κ, Π', Π'' by itself determines F. Construct first the tractors N_κ. Pass planes through each triple of fundamental points, cutting each of the remaining lines in two new points, and giving in all 20 points, four on each line N_κ. Thus with O_κ, A'_κ, A''_κ we have seven points on each of the five lines, and from these we derive 15 anharmonic ratios. These, of course, cannot be independent since they are expressible in terms of eight; but the relations between them are not of sufficient interest to be given here.

COLUMBIA UNIVERSITY, NEW YORK.

ON THE FORMATION OF THE DERIVATIVES OF THE LUNAR COÖRDINATES WITH RESPECT TO THE ELEMENTS*

BY

ERNEST WILLIAM BROWN

In the determination of the action of the planets on the moon, it is usual to use the method of the variation of arbitrary constants. This method requires the knowledge of the derivatives of the lunar coördinates with respect to the angular elements ϵ, π, θ the epochs of mean longitude, perigee and node—and also with respect to the other three elements n, e, γ—the mean motion, eccentricity and sine of the orbital inclination. If we have a theory in which the coördinates, so far as the actions of the sun and moon only are concerned, are expressed literally in terms of these six elements, it is a simple matter to find the derivatives.

But this is practically not the case. The convergence of many of the coefficients when expressed in powers of $m = n/n'$—the ratio of the mean motions—is so slow that a literal theory with the required accuracy seems almost impossible on account of the labor required for its development; the slowness of convergence does not occur as far as powers of e, γ are concerned. The theory of Hansen is entirely numerical and therefore only the derivatives with respect to the angular elements can be obtained from it. The theory† which I hope shortly to bring to a conclusion is semi-numerical, i. e., the numerical value of m is inserted, the other constants being left in a literal form. ‡

In considering how this theory was to be adapted so as to be of use when considering planetary action, the only difficulty was, therefore, the formation of the derivatives with respect to n. It is this difficulty which has been solved in the following pages. The semi-numerical theory once completed, it is shown that the derivatives with respect to n can be made to depend on quadratures with respect to the time. The formulæ are put into a form ready for computation and it is shown how a simple transformation will permit us to use the advantages of canonical sets of constants without being hampered by their disadvantages for purposes of calculation.

* Presented to the Society February 28, 1903. Received for publication April 14, 1903.

† *Theory of the Motion of the Moon*, Memoirs of the Royal Astronomical Society, parts I, II (1897, '99) in vol. 53; part III (1900) in vol. 54. This will be referred to below as T. M. M., parts I, II, III.

‡ A detailed statement will be found in arts. 4–6.

§(i) *Transformation of the integrals of the equations of variations.*

1. Let $x, y, z, \dot{x}, \dot{y}, \dot{z}$, be the coördinates and velocities of a particle moving under the action of a force function F, so that

$$(1) \qquad \frac{d^2x}{dt^2} = \frac{\partial F}{\partial x}, \qquad \frac{d^2y}{dt^2} = \frac{\partial F}{\partial y}, \qquad \frac{d^2z}{dt^2} = \frac{\partial F}{\partial z},$$

the axes being fixed. Denote by α_p $(p = 1, 2, \cdots, 6)$ the arbitrary constants of this motion and put

$$x_p = \frac{dx}{d\alpha_p}, \qquad \dot{x}_p = \frac{d^2x}{dtd\alpha_p} = \frac{d^2x}{d\alpha_p dt} = \frac{d\dot{x}}{d\alpha_p}, \cdots, \cdots,$$

where the solution is supposed to have been found and expressed in terms of the arbitrary constants and the time. It is more convenient at the outset to take the α_p to be *constants* and not some of them to be variables of the form $b_p t + \beta_p$ as is usually done in the applications; certain changes to the latter form will be made later.

The equations of variations are

$$(2) \qquad \ddot{x}_p = \frac{\partial^2 F}{\partial x^2} x_p + \frac{\partial^2 F}{\partial x \partial y} y_p + \frac{\partial^2 F}{\partial x \partial z} z_p, \qquad \ddot{y}_p = \cdots, \qquad \ddot{z}_p = \cdots.$$

They possess the 15 integrals,

$$(3) \quad (p, q) = \dot{x}_p x_q - \dot{x}_q x_p + \dot{y}_p y_q - \dot{y}_q y_p + \dot{z}_p z_q - \dot{z}_q z_p = C_{pq} \quad (p, q = 1, 2, \cdots, 6),$$

where C_{pq} is a constant and $C_{pq} = -C_{qp}$, $C_{pp} \equiv 0$.

Let the solution be such that x, y, z each consists of sines or cosines of sums of multiples of the angles

$$b_j t + \alpha_j \qquad (j = 4, 5, 6),$$

where b_j and the coefficients depend only on the arbitraries $\alpha_1, \alpha_2, \alpha_3$. The solution may also contain other known angles and constants present in F; it is unnecessary to specify these here. Then it is well known that

$$(4) \quad C_{ii'} = 0, \quad C_{jj'} = 0, \quad C_{ij} = \frac{dc_{j-3}}{d\alpha_i}, \quad b_j = -\frac{dB}{dc_{j-3}} \quad (i, i' = 1, 2, 3; j, j' = 4, 5, 6),$$

where c_1, c_2, c_3 are functions of $\alpha_1, \alpha_2, \alpha_3$ and the known constants, and B is a constant expressed in terms of c_1, c_2, c_3 (instead of $\alpha_1, \alpha_2, \alpha_3$) and the known constants.

2. Consider the determinant of the sixth order

$$\Delta = |\dot{x}_p, x_p, \dot{y}_p, y_p, \dot{z}_p, z_p| \qquad (p = 1, 2, \cdots, 6),$$

and form $d\Delta/dt$. The result contains six determinants obtained by differ-

entiating the elements in each of the six columns. The determinants formed by differentiating the elements in the second, fourth and sixth columns will vanish owing to their possession of two identical columns. Those formed by differentiating the elements in the first, third and fifth columns will also vanish when we substitute for $\ddot{x}_p$, $\ddot{y}_p$, $\ddot{z}_p$ their values from equations (2). Hence $d\Delta/dt = 0$ and

$$\Delta = \text{constant} = K.$$

To find K write Δ in the form

$$\Delta = |x_p, -\dot{x}_p, y_p, -\dot{y}_p, z_p, -\dot{z}_p|,$$

and multiply the two forms for Δ by the ordinary rule for the multiplication of determinants. Using the equations (3) it is evident that we obtain

$$\Delta^2 = |C_{pq}| \qquad (p, q = 1, 2, \cdots, 6),$$

Using (4), this gives

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & C_{14} & C_{15} & C_{16} \\ 0 & 0 & 0 & C_{24} & C_{25} & C_{26} \\ 0 & 0 & 0 & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & 0 & 0 & 0 \\ C_{51} & C_{52} & C_{53} & 0 & 0 & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & 0 \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} C_{14} & C_{15} & C_{16} \\ C_{24} & C_{25} & C_{26} \\ C_{34} & C_{35} & C_{36} \end{vmatrix},$$

a comparison of terms showing that the positive square root is to be taken.

Let k_{ij} denote the first minor of C_{ij} in this determinant with the sign so taken that a cyclical interchange of the first three or of the second three suffixes does not require a change of sign in the minor. Then

$$(5) \qquad \Sigma_j k_{ij} C_{i'j} = \Delta \quad \text{or} \quad 0 \quad \text{according as} \quad i' = i \quad \text{or} \quad \neq i.$$

3. Next, denote by Δ_p the first minors of the elements in the first column of Δ and by Δ_{pq} the second minors of those in the first two columns. Then

$$\begin{aligned} \dot{x}_4\Delta_1 &= |\dot{x}_4 x_p, \dot{y}_p, y_p, \dot{z}_p, z_p| && (p = 2, 3, \cdots, 6) \\ &= |x_4\dot{x}_p - C_{4p}, \dot{y}_p, y_p, \dot{z}_p, z_p| && \text{by (3)} \\ &= x_4|\dot{x}_p, \dot{y}_p, y_p, \dot{z}_p, z_p| - C_{42}\Delta_{42} - C_{43}\Delta_{43} && \text{by (4).} \end{aligned}$$

The first determinant in the right hand member is, as before, easily seen to be equal to $d\Delta_1/dt$. Hence, writing down the two similar equations, we have

$$\begin{aligned} \dot{x}_4\Delta_1 - x_4\dot{\Delta}_1 &= -C_{42}\Delta_{42} - C_{43}\Delta_{43}, \\ \dot{x}_5\Delta_1 - x_5\dot{\Delta}_1 &= -C_{52}\Delta_{52} - C_{53}\Delta_{53}, \\ \dot{x}_6\Delta_1 - x_6\dot{\Delta}_1 &= -C_{62}\Delta_{62} - C_{63}\Delta_{63}. \end{aligned}$$

Multiply these equations by k_{14}, k_{15}, k_{16}, respectively and add. The right hand member vanishes by (5) and the left hand member becomes integrable, giving, since we have similar forms for Δ_2, Δ_3,

$$\Delta_i = (k_{i4}x_4 + k_{i5}x_5 + k_{i6}x_6)A_i \qquad (i=1, 2, 3),$$

where A_i is a constant. Similarly

$$\Delta_j = (k_{1j}x_1 + k_{2j}x_2 + k_{3j}x_3)A_j \qquad (j=4, 5, 6).$$

Put

$$X_i = k_{i4}x_4 + k_{i5}x_5 + k_{i6}x_6, \qquad X_j = k_{1j}x_1 + k_{2j}x_2 + k_{3j}x_3.$$

Then

$$\Delta = \Sigma_i A_i \dot{x}_i X_i - \Sigma_j A_j \dot{x}_j X_j \quad (i=1, 2, 3; j=4, 5, 6).$$

To determine the constants A, we notice that if any two suffixes be interchanged in the determinant Δ, it merely changes sign; hence $A_i = A_j = 1$, and

$$\Delta = \Sigma_i \dot{x}_i X_i - \Sigma_j \dot{x}_j X_j,$$

which may also be written

$$\Sigma_i(\dot{x}_i X_i - x_i \dot{X}_i) = K \qquad (i=1, 2, 3).$$

Again, by the properties of determinants,

$$0 = \Sigma x_p \Delta_p = \Sigma y_p \Delta_p = \Sigma y_p \Delta_p = \Sigma z_p \Delta_p = \Sigma \dot{z}_p \Delta_p \quad (p=1, 2, \cdots, 6),$$

so that, if Y_i, Z_i be defined with reference to y_j, z_j in the same way as X_i was with reference to x_j, and if we write down the corresponding expressions for Δ obtained by interchanging x, y, z, we have the following set of 15 equations:

$$(6) \qquad K = \Sigma(\dot{x}_i X_i - x_i \dot{X}_i) = \Sigma(\dot{y}_i Y_i - y_i \dot{Y}_i) = \Sigma(\dot{z}_i Z_i - z_i \dot{Z}_i),$$

$$(7) \qquad \begin{aligned} 0 &= \Sigma(x_i Y_i - y_i X_i) = \Sigma(x_i Z_i - z_i X_i) \\ &= \Sigma(\dot{x}_i Y_i - y_i \dot{X}_i) = \Sigma(\dot{x}_i Z_i - z_i X_i), \end{aligned}$$

with the eight other equations formed by a cyclical interchange between x, y, z; the summations refer in all cases to the values $i = 1, 2, 3$. Of these fifteen equations, not more than ten are easily seen to be independent of one another, whatever be the values given to x_i, y_i, z_i, X_i, Y_i, Z_i.

§ (ii) *Statement of the problem and its formal solution.*

4. The main object of this paper is to determine the derivatives of the coördinates of the moon with respect to the mean motion n, it being supposed that the derivatives with respect to the other arbitraries have been obtained. The results found above are true whenever the coördinates satisfy the specified conditions and the same may be said of most of those which follow; but it will be

convenient to put them into a form which will be applicable directly to the lunar theory, since they are required for the determination of the action of the planets on the moon.

It is well known that the coördinates of the moon referred to fixed axes are expressible by means of the five angles

$$(8)\quad nt+\epsilon, \qquad n't+\epsilon', \qquad cnt+\epsilon-\pi, \qquad gnt+\epsilon-\theta, \qquad n't+\epsilon'-\pi',$$

where n, n' are the mean motions of the sun and moon; $(1-c)n$, $(1-g)n$, those of the perigee and node of the moon; ϵ, π, θ, the usual angular constants of integration referring to the moon; ϵ', π', the known angular constants of the sun's motion. Also, c, g and the coefficients are expansible in powers of

$$\frac{n}{n'}=m, \quad e, \quad \gamma, \quad e', \quad \frac{a}{a'}=\alpha,$$

where e, γ are "eccentricity" and "inclination" constants of integration referring to the moon; e' (eccentricity), a' (mean distance), known constants of the sun's motion; $n^2a^3=$ sum of the masses of the earth and moon; $n'^2a'^3=$ sum of the masses of the earth, moon and sun. The set of arbitrary constants α_1, α_2, α_3 is taken to be n, e, γ and the set α_4, α_5, α_6 to be ϵ, π, θ; n, $n(1-c)$, $n(1-g)$ are the motions of the corresponding angles.

If the coördinates be referred to axes of which one points to the mean place of the sun or moon, the coördinates are functions of *four* angles only, the first two of (8) being replaced by their difference.

5. When it becomes a question of finding the derivatives of the coördinates with respect to the arbitraries, there is no difficulty as far as ϵ, π, θ are concerned, since these only occur under the signs, sine and cosines, whether the theory has been worked out literally or with numerical values for m, e, γ, e', α. But this is not the case with n, e, γ: a literal theory like DELAUNAY's is necessary. But the use of such a theory gives rise to a practical difficulty. Every coefficient of a periodic term is of the form $aA\lambda$ where λ is the product of the highest positive powers of e, γ, e', α which are factors of the coefficient, and A is expansible in powers of m, e^2, γ^2, e'^2, α^2. If we arrange A according to powers of the last four parameters, the convergence is sufficiently rapid for practical purposes and if, further, the coefficient (a function of m only) of each of these powers is determined with sufficient accuracy, there is no difficulty in finding the derivatives with respect to e, γ. It is quite different when we arrange according to powers of m. In the first place, the rate of convergence is in many cases very slow, corresponding frequently to that of a geometric series with a common ratio $\frac{1}{2}$, and the rate of convergence of the earlier terms is less in the series formed by taking the derivatives with respect to n or

m. The most accurate literal theory is that of DELAUNAY and this is insufficient for modern demands. Moreover, it may be said to be almost a practical impossibility to calculate a literal theory to a sufficient degree of approximation.

6. It was with these facts in view that I set out to obtain a semi-numerical theory in which the parameters e, e', γ, α were left in their literal form, while the numerical value of m (which is known from observation with very high accuracy) was used from the start. The values of the coördinates have been published* as far as the fourth order inclusive in these four parameters and this order is probably sufficiently high to obtain the derivatives of the coördinates with respect to e, γ with the degree of accuracy required to obtain the action of the planets on the moon. But we cannot, of course, directly obtain from it the derivatives with respect to m. It might be possible to use DELAUNAY's theory in conjunction with the numerical results by a somewhat empirical method which I adopted in an earlier paper† and which served well the purpose for which it was intended. But it appears doubtful whether it would be sufficiently accurate for the determination of all the planetary inequalities. I have therefore considered this problem:

Given the derivatives of the various functions with respect to ϵ, π, θ, e, γ, *to find those with respect to* n *from a theory in which the numerical value of* m *has been substituted.*

The possibility of solving this problem practically arose from considering a subsidiary result which I gave in an earlier paper,‡ where it was solved so far as the terms independent of e, e', γ, α were concerned; the object and the method were, however, quite different from those of this paper. In the following sections the problem will first be solved in the form which naturally presents itself and afterwards the results will be so transformed that certain difficulties arising from the infinities of the integrals are avoided and that the results of the theory, above referred to, can be directly used.

7. It has just been seen that we require to determine one set of the derivatives in terms of the other five: let this set be x_1, y_1, z_1. For this purpose we have the fifteen equations (6), (7); we have seen, however, that they are not all independent. Let us attempt to determine x_1 without an integration from

$$\Sigma(\dot{x}_i X_i - x_i \dot{X}_i) = K, \quad \Sigma(x_i Y_i - y_i X_i) = 0, \quad \Sigma(\dot{x}_i Y_i - y_i \dot{X}_i) = 0,$$

of which the terms containing the suffix 1 are

$$\dot{x}_1 X_1 - x_1 \dot{X}_1, \qquad x_1 Y_1 - y_1 X_1, \qquad \dot{x}_1 Y_1 - y_1 \dot{X}_1,$$

* T. M. M., parts I, II, III.

† *On the theoretical values of the secular accelerations in the lunar theory*, Monthly Notices of the Royal Astronomical Society, vol. 57 (1897), p. 346.

‡ *On the solution of a pair of simultaneous differential equations*, etc., Memoirs of the Cambridge Philosophical Society, vol. 18 (Stokes Memorial, 1900), pp. 94–106.

and the capital letters contain the suffices 4, 5, 6 only. The Jacobian of these with respect to $\dot{x}_1$, x_1, y_1 is zero; a similar result appears to hold if we attempt to combine any of the equations. Hence we shall use the first two of (6) to determine x_1, y_1, and one of the others to determine z_1. We find

$$x_1 = X_1 \int \frac{K - \dot{x}_2 X_2 + x_2 \dot{X}_2 - \dot{x}_3 X_3 + x_3 \dot{X}_3}{X_1^2} dt, \tag{9}$$

with a similar equation for y_1; no added constant is necessary since x_1 cannot contain terms of the form AX_1, that is, of the form Ax_j, for the x_j consist of sines and x_1 of cosines. It will be sufficient to consider the equation for x_1 since that for y_1 is exactly similar; the equation for z_1 is treated separately.

8. There are three points to be considered in connection with the equation (9). First, x_2, x_3, $\dot{x}_2$, $\dot{x}_3$ give rise to terms containing the time as a factor owing to the presence of e, γ in c, g, that is, in the arguments of the periodic terms; practical convenience requires this to be avoided. Second, X_2, X_3 contain the derivatives dc_i/dn owing to their presence in certain of the coefficients k_{ij}; these must either be calculated or eliminated. Third, the principal term of X_1, which appears as a denominator under the integral sign, is the derivative of $a \cos(nt + \epsilon)$ with respect to n and it may therefore vanish. In order to show how the first two difficulties may be avoided I shall develop the results with a canonical system of constants and then adapt them for practical use by the adoption of a certain semi-canonical system. The third difficulty easily disappears when complex coördinates are used.

9. Instead of $\alpha_1 = n$, $\alpha_2 = e$, $\alpha_3 = \gamma$ as the first set of arbitrary constants, let us take $\alpha_1 = c_1$, $\alpha_2 = c_2$, $\alpha_3 = c_3$, and suppose that it be required to find the derivatives with respect to c_1, those with respect to c_2, c_3, ϵ, π, θ being supposed known. We have

$$K = 1, \qquad k_{i,i+3} = 1, \qquad k_{ij} = 0 \qquad (j \neq i+3),$$
$$X_i = x_{i+3}, \qquad Y_i = y_{i+3}, \qquad Z_i = z_{i+3};$$

and the first of equations (6) becomes

$$\Sigma_i (\dot{x}_i x_{i+3} - x_i \dot{x}_{i+3}) = 1 \qquad (i = 1, 2, 3). \tag{10}$$

Denote the three angles by $w_j = b_j t + \beta_j$ $(j = 4, 5, 6)$. Then

$$\frac{dx}{dc_i} = \frac{\partial x}{\partial c_i} + t \Sigma_j \frac{db_j}{dc_i} x_j,$$

$$\frac{d\dot{x}}{dc_i} = \frac{d}{dt}\frac{\partial x}{\partial c_i} + \left(1 + t\frac{d}{dt}\right)\left(\Sigma_j \frac{db_j}{dc_i} x_j\right) = \frac{\partial}{\partial c_i}\frac{dx}{dt} + t\frac{d}{dt}\left(\Sigma_j \frac{db_j}{dc_i} x_i\right),$$

where $\partial/\partial c_i$ denotes that the derivative is taken with respect to c_i, only in so far as c_i occurs in the *coefficients* of the periodic terms of the function on which it operates.

Substitute in (10) and equate separately to zero the terms which do and do not contain t as a factor. The latter give the well known results

$$\frac{db_j}{dc_i} = \frac{db_{i+3}}{dc_{j-3}} \qquad (i=1,2,3;\ j=4,5,6).$$

The former give

$$\Sigma_i\left(x_{i+3}\frac{\partial \dot{x}}{\partial c_i} - \dot{x}_{i+3}\frac{\partial \dot{x}}{\partial c_i}\right) = 1,$$

or

$$\Sigma_i\left(x_{i+3}\frac{d}{dt}x_i - \dot{x}_{i+3}x_i\right) + \Sigma_j\Sigma_i\frac{db_j}{dc_i}x_j x_{i+3} = 1,$$

according as we use the second or first form of $d\dot{x}/dc_i$. It is necessary to use the first form for $i=1$ on account of the integration with respect to t and it is better to use the second form for $i=2, 3$. The equation then becomes

$$x_4\frac{d}{dt}\frac{\partial x}{\partial c_1} - \dot{x}_4\frac{\partial x}{\partial c_1} = 1 - x_5\frac{\partial \dot{x}}{\partial c_2} + \dot{x}_5\frac{\partial x}{\partial c_2} - x_6\frac{\partial \dot{x}}{\partial c_3} + \dot{x}_6\frac{\partial x}{\partial c_3} - x_4\Sigma_j\frac{db_j}{dc_1}x_j$$

$$= Q - x_4^2\frac{db_4}{dc_1},$$

where the definition of Q is evident.

Dividing by x_4^2, putting $db_5/dc_1 = db_4/dc_2$, $db_6/dc_1 = db_4/dc_3$, and integrating, we obtain

$$\frac{\partial x}{\partial c_1} = x_4\int\left(\frac{Q}{x_4^2} - \frac{db_4}{dc_1}\right)dt.$$

The only derivative with respect to c_1 in the right hand member is db_4/dc_1. But since $\partial x/\partial c_1$ can not contain any term factored by t, the constant term under the integral sign must vanish. This fact gives

$$\frac{db_4}{dc_1} = \text{const. term in the expansion of } \frac{Q}{x_4^2},$$

and the determination of $\partial x/\partial c_1$ is complete.

10. The objection to leaving the results in this form for practical application is the fact that the formation of the coördinates as functions of c_1, c_2, c_3 is very troublesome, especially as far as c_1 is concerned. But I shall show later that there is no objection to the use of c_2, c_3 instead of e, γ. Hence it will be convenient to develop the results for the *semi-canonical* system $b_4 (= n)$, c_2, c_3, so that c_1 is considered to be a function of the independent constants n, c_2, c_3.

With this semi-canonical system, we find

$$C_{14} = \frac{dc_1}{dn}, \quad C_{24} = \frac{dc_2}{dn}, \quad C_{34} = \frac{dc_3}{dn}, \quad \frac{db_5}{dn} = -\frac{dc_1}{dc_2}, \quad \frac{db_6}{dn} = -\frac{dc_1}{dc_3},$$

$$1 = C_{25} = C_{36} = k_{14}, \qquad 0 = C_{15} = C_{35} = C_{16} = C_{26} = k_{24} = k_{26} = k_{34} = k_{35},$$

$$k_{25} = k_{36} = K = \frac{dc_1}{dn}, \qquad k_{15} = -\frac{dc_1}{dc_2}, \qquad k_{16} = -\frac{dc_1}{dc_3};$$

$$X_1 = x_4 - \frac{dc_1}{dc_2}x_5 - \frac{dc_1}{dc_3}x_6, \qquad X_2 = \frac{dc_1}{dn}x_5, \qquad X_3 = \frac{dc_1}{dn}x_6;$$

$$\frac{dx}{dn} = \frac{\partial x}{\partial n} + t\left(x_4 + \frac{db_5}{dn}x_5 + \frac{db_6}{dn}x_6\right) = \frac{\partial x}{\partial n} + tX_1,$$

$$\frac{d\dot{x}}{dn} = \frac{d}{dt}\frac{\partial x}{\partial n} + \dot{X}_1 + t\dot{X}_1,$$

and the new form of equation (9) is

$$\frac{\partial x}{\partial n} = X_1 \int \left(\frac{dc_1}{dn}\frac{Q'}{X_1^2} - 1\right) dt \tag{11}$$

where X_1 has the value just given and

$$Q' = 1 - x_5\frac{\partial \dot{x}}{\partial c_2} + \dot{x}_5\frac{\partial x}{\partial c_2} - x_6\frac{\partial \dot{x}}{\partial c_3} + \dot{x}_6\frac{\partial x}{\partial c_3}.$$

There are no derivatives with respect to n in (11) except dc_1/dn and this is determined by the vanishing of the constant term under the integral sign. It is remarkable that the expression for Q' which arises with the semi-canonical system is shorter than that for Q which arises with the canonical system. The equation (11) might have been obtained from (10) by direct transformation of the set c_1, c_2, c_3 to the set n, c_2, c_3.

11. There are two determinations to be made. The first is to find the values of c_1, c_2, c_3 from a theory expressed in terms of n, e, γ and the second is to express this theory as well as c_1 in terms of n_1, c_2, c_3.

For the first, I have shown * that

$$c_i = \text{const. term in } \dot{x}x_{i+3} + \dot{y}y_{i+3} + \dot{z}z_{i+3},$$

which permits of the calculation of c_i with ease. We find

$$c_1 = \sqrt{a\mu}P_1(e^2, \gamma^2), \quad c_2 = e^2\sqrt{a\mu}P_2(e^2, \gamma^2), \quad c_3 = \gamma^2\sqrt{a\mu}P_3(e^2, \gamma^2),$$

* *On certain Properties of the Mean Motions*, etc., Proceedings of the London Mathematical Society, vol. 28 (1896), p. 150. The proof involves the supposition $a/a' = 0$. That the result holds when $a/a' \neq 0$ will be shown in a paper to appear later in the Transactions.

where P_1, P_2, P_3 are positive power series in e^2, γ^2, and μ is the sum of the masses of the earth and moon. It is understood that the numerical values of the other constants may have been substituted: it is unnecessary in any case to specify them. From the second and third of these equations, the values of e^2, γ^2 are quickly deduced as power series in $c_2/\sqrt{a\mu}$, $c_3/\sqrt{a\mu}$, particularly as it will probably never be necessary to go beyond the fourth powers of e, γ. Hence c_1 is easily expressible in powers of the same quantities and likewise b_5, b_6.

The coefficients contain odd powers of e, γ in general and therefore will involve powers of the square roots of the new parameters. The final transformation is therefore to put

$$c_2 = \mathrm{E}^2\sqrt{a\mu} = \mathrm{E}^2 a^2 n, \qquad c_3 = \Gamma^2\sqrt{a\mu} = \Gamma^2 a^2 n,$$

so that in Q or Q' we can put

$$\frac{\partial}{\partial c_2} = \frac{1}{2a^2 n\mathrm{E}}\frac{\partial}{\partial \mathrm{E}}, \qquad \frac{\partial}{\partial c_3} = \frac{1}{2a^2 n\Gamma}\frac{\partial}{\partial \Gamma},$$

where the coördinates are transformed by the equations

$$e = \mathrm{E}P_2(\mathrm{E}^2, \Gamma^2), \qquad \nu = \Gamma P_3(\mathrm{E}^2, \Gamma^2).$$

It is to be noticed that the value of z_1 may be obtained without the necessity of an integration from either of the equations

$$\Sigma_i(x_i Z_i - z_i X_i) = 0 = \Sigma_i(y_i Z_i - z_i Y_i), \tag{12}$$

when x_1, y_1 have been obtained.

12. We have next to transform to complex coördinates to avoid the occurrence of vanishing denominators. Put

$$u = x + y\sqrt{-1}, \qquad u_i = x_i + y_i\sqrt{-1}, \qquad U_i = X_i + Y_i\sqrt{-1},$$
$$s = x - y\sqrt{-1}, \qquad s_i = x_i - y_i\sqrt{-1}, \qquad S_i = X_i - Y_i\sqrt{-1}.$$

From the equations (6), (7) we easily obtain

$$0 = \Sigma_i(\dot{u}_i U_i - u_i \dot{U}_i) = \Sigma_i(u_i Z_i - z_i U_i), \tag{13}$$

$$\Sigma_i(\dot{u}_i S_i - s_i \dot{U}_i) = 2K, \qquad \Sigma_i(u_i S_i - s_i U_i) = \Sigma_i(\dot{u}_i Z_i - z_i \dot{U}_i) = 0, \tag{14}$$

with the corresponding equations formed by changing the sign of $\sqrt{-1}$. The first of (13) determines u_1, with the denominator U_1^2 which does not vanish *; the second equation then gives z_1. The other equations may be used for verification purposes.

*Since X_1, Y_1 never vanish simultaneously.

It is quite evident that, in order to obtain the equations corresponding to those of arts. 7–11, all that is necessary is to substitute u for x, U for X, and omit the term unity in Q and Q'. To find $\partial z/\partial u$ we have

$$\frac{\partial z}{\partial u} = \frac{u_1}{U_1} Z_1 - \frac{1}{U_1}\frac{dc_1}{dn}\left(\frac{\partial u}{\partial c_2} z_5 - \frac{\partial z}{\partial c_2} u_5 + \frac{\partial u}{\partial c_3} z_6 - \frac{\partial z}{\partial c_3} u_6\right), \tag{15}$$

where

$$U_1 = u_4 - \frac{dc_1}{dc_2} u_5 - \frac{dc_1}{dc_3} u_6, \qquad Z_1 = z_4 - \frac{dc_1}{dc_2} z_5 - \frac{dc_1}{dc_3} z_6.$$

13. As a rule it is more convenient to have the coördinates referred to axes of which one is directed to the mean place of the sun or moon. There is no trouble in either case in making the change. We substitute

$$u \exp. + \sqrt{-1}\,(n't + \epsilon') \quad \text{or} \quad u \exp. + \sqrt{-1}\,(nt + \epsilon),$$

in the respective cases, for u. The various small charges of this nature which experience has shown to be best for actual calculation are made in the summary which follows. It is scarcely useful to give all the details of the algebraical work which leads up to the results. The formulæ are somewhat long and necessarily unsymmetrical, as is generally the case when they are best adapted for numerical computation. It is to be noted also that the calculations are of the same nature as those which constitute the bulk of the work in my lunar theory, namely, multiplications of series. The great majority of these multiplications are, however, already done, having been previously needed, so that the new work is much less extensive than would appear from the formulæ. The order of accuracy with respect to e, γ to which they give the values of the derivatives with respect to n is one less than that of the coördinates, but in any case it can be found to a sufficient degree of accuracy for practical purposes.

§ (iii) *Summary of the results adapted for calculation.*

14. The notation adopted in this section is that of my theory.* It is therefore necessary to substitute for the symbols denoted above by u, s, the values

$$u \exp. \sqrt{-1}\,(n't + \epsilon'), \qquad s \exp. - \sqrt{-1}\,(n't + \epsilon')$$

to reduce to axes moving with the sun's mean motion. We shall suppose that this substitution has been made.

Now u is expansible in powers of ζ, ζ^e, ζ^z, ζ^m. Let

$$D_1 = \zeta^1 \frac{\partial}{\partial \zeta^1}, \qquad D_e = \zeta^e \frac{\partial}{\partial \zeta^e}, \qquad D_g = \zeta^g \frac{\partial}{\partial \zeta^g}, \qquad D_m = \zeta^m \frac{\partial}{\partial \zeta^m},$$

* T. M. M., part I.

a notation which can be adopted since c, g, m, are incommensurable with unity and their numerical values are not substituted in the index of ζ. Then

$$D = D_1 + cD_c + gD_g + mD_m.$$

Put*

$$au' = u\zeta^{-1}. \qquad as' = s\zeta, \qquad az' = z;$$

then $u'a$ is the complex coördinate referred to the mean place of the moon as axis of x.

15. I shall now indicate briefly the various steps required to find the derivatives with respect to six arbitraries.

(a) The derivatives with respect to ϵ, π, θ are given by

$$(A) \qquad \frac{du'}{d\epsilon} = (D_1 + D_c + D_g)u'\sqrt{-1},$$

$$\frac{du'}{d\pi} = -D_c u'\sqrt{-1}, \qquad \frac{du'}{d\theta} = -D_g u'\sqrt{-1}.$$

(b) The canonical constants c_1, c_2, c_3 are given by the method referred to in art. 11. The equations for them and for certain new constants (not quite the same as those of art. 11) introduced for convenience, are

$$\begin{aligned}(n-n')a^2 A = c_1 = -\tfrac{1}{2}(n-n')a^2\big[&(D+1+m)u' \cdot (D_1+D_c+D_g-1)s' \\ &+ (D-1-m)s' \cdot (D_1+D_c+D_g+1)u' \\ &+ 2Dz' \cdot (D_1+D_c+D_g)z'\big]^\circ,\end{aligned}$$

$$(B) \qquad \begin{aligned}-(n-n')a^2 E^2 = c_2 = \tfrac{1}{2}(n-n')a^2\big[&(D+1+m)u' \cdot D_c s' \\ &+ (D-1-m)s' \cdot D_c u' + 2Dz' \cdot D_c z'\big]^\circ,\end{aligned}$$

$$\begin{aligned}-(n-n')a^2 \Gamma^2 = c_3 = \tfrac{1}{2}(n-n')a^2\big[&(D+1+m)u' \cdot D_g s' \\ &+ (D-1-m)s' \cdot D_g u' + 2Dz' \cdot D_g z'\big]^\circ,\end{aligned}$$

where $[\ ]^\circ$ denotes the constant term in the development of the enclosed function and μ is put equal to unity. The actual calculation from these rather long formulæ is really quite brief. The operators D_1, D_c, D_g introduce only integers and the necessary multiplications of series are nearly all at hand; with these, a few hours' work will suffice to determine c_1, c_2, c_3 to the fourth order.†

We thus find A, E^2, Γ^2 expanded in powers of e^2, k^2 (the constants of my

* The symbol u' is not used in T. M. M.

† The relations of c_i to DELAUNAY's final L, G, H, are given by

$$c_1 = L, \qquad c_2 = (G - L), \qquad c_3 = (H - G),$$

when $\mu = 1$.

theory) with numerical coefficients if the numerical values of e', α have been substituted, and in powers of e'^2, α^2 if their numerical values have not been substituted.

(*c*) Revert the series for E^2, Γ^2 so as to find e^2, k^2 expressed in powers of E^2, Γ^2. Then find A, e, g, in terms of the latter constants. Take the positive square roots to obtain e, k.

(*d*) Substitute the values of e, k in the expressions for u', z', so as to obtain the coördinates in terms of E, Γ.

(*e*) Put

$$U' = (D_1 + D_e + D_g + 1)\,u' - \frac{d_A}{d\mathrm{E}^2} D_e u' - \frac{d_A}{d\Gamma^2} D_g u',$$

$$Q = \frac{D_e u'}{\mathrm{E}} \frac{\partial}{\partial \mathrm{E}} Du' - \frac{D_e Du'}{\mathrm{E}} \frac{\partial u'}{\partial \mathrm{E}} + 2D_g u' \frac{\partial}{\partial \Gamma^2} D_g u' - 2D_g Du' \frac{\partial u'}{\partial \Gamma^2},$$

$$\frac{1}{\beta} = \left[\frac{Q}{2U'^2}\right]^\circ, \qquad \frac{dc_1}{dn} = -\mathrm{a}^2\beta,$$

and compute U', Q and the constant β. Then we find the derivative with respect to n from the formula

$$(C) \qquad \frac{n - n'}{\mathrm{a}} \frac{\partial (u'\mathrm{a})}{\partial n} = U' D^{-1}\left(\frac{\beta Q}{2U'^2} - 1\right),$$

no added constant being necessary on performing the operation D^{-1}. The operations $\partial/\partial \mathrm{E}$, $\partial/\partial \Gamma$ are performed only on the *coefficients* of the various powers of ζ in u', Du'. The final result is the value of $\partial(u'\mathrm{a})/\partial n$ when E, Γ have been replaced by their values in terms of c_2, c_3: the change was a temporary one for convenience only.

The actual calculation of Q/U'^2 is performed in the following way. The principal terms in u' (those depending only on m and therefore of the form $\Sigma_i a_i \zeta^{2i}$ where the a_i are numerical coefficients and $a_0 = 1$) being denoted by u'_0, the principal terms in U' are given by

$$U'_0 = (D_1 + 1)\,u'_0 = (D + 1)\,u'_0.$$

Hence

$$\frac{1}{U'^2} = \frac{1}{U'^2}\left\{1 - 2\frac{U' - U'_0}{U'_0} + 3\left(\frac{U' - U'_0}{U'_0}\right)^2 - \cdots\right\}.$$

The functions $1/U'^p_0$, being power series in ζ^2 are best calculated by the method of special values. It is to be noted that all terms on which D_e has operated contain the factor E, and those on which D_g has operated, the factor Γ^2.

In the expansion of $1/U'^2$, the value of u' is used as far as the order actually required; in that of Q, the value of u' to one order higher in E, Γ.

The values of $\partial u'/\partial \mathrm{E}$, $\partial u'/\partial \Gamma$ have been obtained in the course of the investigation and we therefore have the derivatives of with respect to all six constants.

(f) Finally separate into real and imaginary parts and obtain the derivatives of the two coördinates referred to moving axes, that of x pointing to the mean place of the moon. For the transformation we have

$$(D)\quad \begin{aligned} \zeta^{i} &= \cos iD + \sqrt{-1}\sin iD, & \zeta^{ic} &= \cos il + \sqrt{-1}\sin il, \\ \zeta^{ig} &= \cos iF + \sqrt{-1}\sin iF, & \zeta^{im} &= \cos il' + \sqrt{-1}\sin il', \end{aligned}$$

where D, l, l', F are the usual angles in DELAUNAY's notation.

(g) Put

$$Z' = (D_1 + D_c + D_g)z' - \frac{d_A}{d\mathrm{E}^2} D_c z' - \frac{d_A}{d\Gamma^2} D_g z',$$

and determine $\partial z/\partial n$ from

$$(E)\qquad \frac{n - n'}{\mathrm{a}} \frac{\partial z}{\partial n} = Z' D^{-1}\left(\frac{\beta Q}{2U'^2} - 1\right) + \frac{\beta}{2U'} Q',$$

where

$$Q' = \frac{\partial u'}{\partial \mathrm{E}} \cdot \frac{D_g z'}{\mathrm{E}} - \frac{\partial z'}{\partial \mathrm{E}} \cdot \frac{D_c u'}{\mathrm{E}} + 2\frac{\partial u'}{\partial \Gamma^2} D_g z' - \frac{\partial z'}{\partial \Gamma} \frac{D_g u'}{\Gamma}.$$

The expansion of $1/U'$ is made as before and the calculation of the various functions follows a quite similar plan.

16. The value of $\partial(u_0'\mathrm{a})/\partial n$ has already been obtained. If the suffix zero denotes the parts depending only on m, the connection between the results here and those in T. M. M., pt. II, chap. V, section (ii), is given by

$$U_0' = \mathrm{u}_3', \qquad n\frac{\partial(u_0'\mathrm{a})}{\partial n} = 2\mathrm{u}_4\mathrm{a}\beta_0,$$

$$-\beta_0 = \frac{1}{\mathrm{a}^2}\frac{d}{dn}(c_1)_0 = -\frac{1}{2\mathrm{q}} = -.33022, \qquad \frac{d}{dn}(c_1)_0 = -\frac{\mathrm{a}^2}{2\mathrm{q}} = -.32962a^2.$$

I have found elsewhere* by a totally different method,

$$\frac{d}{dn}(c_1)_0 = \tfrac{3}{2}\frac{d^2}{dn^2}\left[\frac{\mu}{r}\right]_0^\circ, \qquad \frac{d^2}{dn^2}\left[\frac{\mu}{r}\right]_0^\circ = -.21975a^2,$$

which gives the same value for $d(c_1)_0/dn$.

* Page 153 of the paper referred to in art. 11 and page 348 of the first paper referred to in art. 6.

18. If it be intended actually to use the canonical equations for the variation of arbitrary constants, namely,

$$\frac{dc_i}{dt} = \frac{\partial R}{\partial w_{i+3}}, \qquad \frac{dw_j}{dt} = -\frac{\partial R}{\partial c_{j-3}} + b_j \qquad (i=1, 2, 3; j=4, 5, 6),$$

it is easy to make the necessary changes. Denote by parentheses the derivatives when c_1, c_2, c_3 are considered to be the arbitraries. Then if ϕ be any one of the functions used, the derivatives of ϕ with regard to c_1, c_2, c_3 are found by solving the equations

$$\frac{\partial \phi}{\partial n} = \left(\frac{\partial \phi}{\partial c_1}\right)\frac{dc_1}{dn}, \quad \frac{\partial \phi}{\partial c_2} = \left(\frac{\partial \phi}{\partial c_1}\right)\frac{dc_1}{dc_2} + \left(\frac{\partial \phi}{\partial c_2}\right), \quad \frac{\partial \phi}{\partial c_3} = \left(\frac{\partial \phi}{\partial c_1}\right)\frac{dc_1}{dc_3} + \left(\frac{\partial \phi}{\partial c_3}\right),$$

with respect to $(\partial\phi/\partial c_1)$, $(\partial\phi/\partial c_2)$, $(\partial\phi/\partial c_3)$; $dc_1/dn = -\beta a^2$ is given by art. 15 (e) and dc_1/dc_2, dc_1/dc_3 will have been found.

HAVERFORD COLLEGE,
April 14, 1903.

ON REDUCIBLE GROUPS*

BY

SAUL EPSTEEN

In the Transactions of the American Mathematical Society, January, 1903, pages 46–56 Professor LOEWY proves for the first time the theorem: "*Wie auch immer eine Gruppe G linearer homogener Substitutionen unter Hervorhebung ihrer irreduciblen Bestandteile in eine ähnliche Gruppe transformirt wird, so kann man die irreduciblen Bestandteilen, die sich bei irgend einer Darstellung ergeben, den irreduciblen Bestandteilen die sich bei irgend einer anderen Darstellung ergeben, eineindeutig so zuordnen, dass zwei zugeordnete irreducible Teilgruppen gleich viele Variablen haben und ähnliche Gruppen sind.*"

This says that if by transformation G be decomposed into

$$(1)\qquad \begin{matrix} \alpha_{11} & 0 & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{\lambda 1} & \alpha_{\lambda 2} & \alpha_{\lambda 3} & \cdots & \alpha_{\lambda\lambda} \end{matrix}$$

and

$$(2)\qquad \begin{matrix} \beta_{11} & 0 & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_{\mu 1} & \beta_{\mu 2} & \beta_{\mu 3} & \cdots & \beta_{\mu\mu} \end{matrix}$$

the irreducible groups

$$\begin{matrix} \alpha_{11} & \alpha_{22} & \cdots & \alpha_{\lambda\lambda} \\ \beta_{11} & \beta_{22} & \cdots & \beta_{\mu\mu} \end{matrix}$$

are similar to each other in *some* order, that the similar group α_{ii} and β_{jj} have the same number of variables, and that $\lambda = \mu$.

The object of this note is to prove the same theorem more briefly. It is well to notice however that LOEWY's proof has the advantage of being direct, while mine, although much shorter, is indirect.

Suppose we have before us a reducible linear homogeneous group G in n variables, then there exists a system of linear differential equations which have

* Presented to the Society February 28, 1903. Received for publication February 21, 1903.

G for their group of rationality.* These equations are all of the same order and of the same *type.*† Let $P_n y = 0$ be some one of these linear differential equations whose group of rationality is G. Since by hypothesis G is reducible, $P_n y = 0$ is also reducible.‡ We now decompose P_n into irreducible factors; this may of course be accomplished in various ways:

$$P_n = Q_\lambda Q_{\lambda-1} \cdots Q_2 Q_1 = 0, \tag{3}$$

$$P_n = R_\mu R_{\mu-1} \cdots R_2 R_1 = 0. \tag{4}$$

To these decompositions of the equation there are corresponding decompositions of G, namely (1) and (2), where α_{ii} is the group of $Q_i = 0$ $(i = 1 \cdots \lambda)$ and β_{jj} is the group of $R_j = 0$ $(j = 1 \cdots \mu)$.

Now LOEWY showed in his paper entitled *Ueber reduzible lineare homogene Differentialgleichungen* § for the decompositions (3) and (4) that $\lambda = \mu$, and that, for suitable values of i and j, Q_i and R_j are of the same *order* and *type.*

Since $\lambda = \mu$ it follows that the number of irreducible groups α equals the number of irreducible groups β. Since Q_i and R_j are of the same order, the number of variables in α_{ii} is the same as the number of variables in β_{jj} (the number of variables in the group being the same as the order of the differential equation). Finally, since Q_i and R_j are of the same type, α_{ii} and β_{jj} must be similar.‖

The theorem may be stated in the following manner:** *Considering similar groups as equivalent, the irreducible constituents of a linear homogeneous group G which are obtained by the various possible decompositions, are equivalent apart from the arrangement.*

THE UNIVERSITY OF CHICAGO.

February 20, 1903.

---

* JORDAN, Bulletin de la Société Mathematique de France, vol. 2, p. 104; KLEIN, *Höhere Geometrie*, II, pp. 296–7; LOEWY, Mathematische Annalen, January, 1903, p. 560. For this existence theorem, see SCHLESINGER, *Handbuch* II, 1, pp. 76–7, 93.

† The French *espèce* might be translated *species*, and the German *Art* as *kind*. But in LOEWY's paper in the Annalen of January, 1903, p. 560, it appears that the groups of these equations are all of the same type. For this reason I say that the *equations are of the same type.* In accordance with the customary practice, LOEWY calls such groups equivalent.

‡ SCHLESINGER, *Handbuch*, II, 1, p. 106.

§ Mathematische Annalen, January, 1903, p. 565.

‖ According to SCHLESINGER's *Handbuch*, II, 1, p. 121, α_{ii} and β_{jj} are the same; but SCHLESINGER means by this that they are of the same type, i. e., can be transformed into each other. In this connection, cf. VESSIOT, Annales de l'École Normale Supérieure, 1892, p. 232.

** Cf. LOEWY, Transactions, January, 1903, p. 47.

THEORY OF LINEAR ASSOCIATIVE ALGEBRA*

BY

JAMES BYRNIE SHAW

Introduction.

The following paper is a development of *Linear Associative Algebra*, as distinguished from linear associative algebras. That is to say, not only are algebraic systems of a definite number of units handled, but the foundation is laid for a treatment of associative numbers in general, irrespective of whether they belong to a system of units of one order rather than of another. From the writer's standpoint, a hypercomplex number and an associative number are extensions of the field of number beyond the limits of ordinary negative or complex numbers. An associative number is a number defined by some equation, or set of equations, whose terms and their component factors are *associative* quantities. Thus, a *quaternion* may be *defined* by the identity

$$q^2 - 2wq + \Sigma t_i^2 \equiv 0,$$

where w and $t_1, t_2 \cdots t_r$ are "scalars," which means that wherever q^2 occurs we might substitute for it

$$2wq - \Sigma t_i^2.$$

From this point of view we should define a radical not by a series, or a "sequence," but by the expression which may be substituted for a term of higher degree, as, in defining θ by the equation (or "identity")

$$\theta^2 \equiv 2.$$

The equation above for q contains *essentially* all the properties of quaternions (at least of real quaternions) and from this point of view the fact that quaternions belong to an algebra of *four* units is of minor importance. It was BENJAMIN PEIRCE who said regarding the classification by "order"†:

"This artificial division of the algebras is cold and uninstructive like the artificial Linnean system of botany."†

*Presented to the Society at the Evanston meeting, September 2-3, 1902, under a slightly different title. Received for publication September 11, 1902.

†PEIRCE, *Linear Associative Algebra*, American Journal of Mathematics, vol. 4 (1884), p. 99.

The first result of this paper is to find an answer to the question: *What are the possible independent numbers defined by an identity*

$$q^n - w_1 q^{n-1} + w_2 q^{n-2} - \cdots \pm w_n \equiv 0,$$

where the only limitation on q is associativity, and where $w_1, w_2, \cdots, w_n$ are homogeneous expressions of degrees $1, 2, \cdots, n$ dependent on arbitrary scalars $x_1, x_2, \cdots, x_m$. The explicit answer to this question is to be found in Part 3.

The second result of importance is to establish a *calculus* of linear associative algebra. This calculus depends on the notation for its power and simplicity, and the notation depends on the conception of certain "associative units." It is proved that every associative number is expressible in a form built up by adding together terms, each of which consists of one of these associative units with a coefficient dependent linearly and homogeneously on the arbitrary scalar coefficients of the associative number. These expressions may be built and used without knowing explicitly how many arbitrary scalar coefficients the coefficients of the associative units depend on. There is more than a mere notation involved here, and this is more than an expression of results in a matricular notation. Just as the "quadrate or relative *vids*" of C. S. PEIRCE * enable matrices to be conceived as the sum of n^2 terms, each with definite laws of combination with the others, so these associative units of form λ_{ijk} enable us to conceive all associative numbers as really expressions linear in these units, each with definite and simple laws of combination with the others. And as all matrices are expressible in terms of vids and the theory of matrices is essentially the theory of vids, so the theory of linear associative algebra is the theory of these associative units. From this point of view the matrix becomes a special case of linear associative algebra, and an associative number obeys laws different in some respects from those of matrices. We regard all associative numbers as belonging to an associative algebra of an infinite number of units.

The third result of importance is that these results are independent of the presence or absence of a modulus. Any result based on the group-theory of LIE is dependent on the presence of the "identical transformation," *i. e.*, there must be a *modulus* in the algebra. It is obvious that every such result must needs be shown separately to apply to algebras (by far the more numerous) not including a modulus. It was BENJAMIN PEIRCE who recognized intuitively the great importance of these *nilpotent* algebras.† In fact, the problem of writing all nilpotent algebras and of developing nilpotent numbers, is almost commensurate

* PEIRCE, *Linear Associative Algebra*, American Journal of Mathematics, vol. 4, p. 221.

† PEIRCE, American Journal of Mathematics, vol. 4, p. 118.

with developing the entire field. These numbers satisfy each an identity of the form

$$q^m \equiv 0.$$

They are not determinable from the forms satisfying the complete equation cited above, nor are their properties to be developed from such an equation. This fact is the real basis of Hamilton's refusal to admit "imaginary quaternions" into his system. They belong to a class of number different from the real quaternions. The problem of nilpotents is one on which little has been done, save incidentally in the study of algebras with an idempotent unit. The present paper produces general forms for all nilpotent associative numbers and the combinations of these numbers may be studied aside from their belonging to any limited algebra. A subsequent paper presents them in greater detail.

The fourth result of importance is that the study of algebras of any certain order is made easy. All algebras of order n may be found with little trouble, and by comparatively rapid methods. Furthermore it is not necessary that all algebras of order $n-1$ be known, for the entire list of algebras of order n may be found without knowing those of order $n-1$. The natural transformations and simplifications of the algebra are further evident from the forms of the units, and no multiplication table is needed, as the combination laws are evident from the forms of the units. Any algebra may readily be thrown into this form.

The fifth result is the great simplification of proofs and processes from this standpoint of wider generality (which would be expected). The various systems of algebras that belong together on different bases of classification are readily seen, and many new bases of classification are suggested.

This paper may be looked upon as an introduction to the subject, in which only the fundamental principles are laid down. It is hoped that the field will be developed farther in the near future.

An analytical table of the sections is given to serve as a guide in the reading.

Part 1. The Matrix form of a Linear Algebra.
§ 1. Definition of product.
§ 2. Matrix form for product, viewing the facient as operator.
§ 3. Matrix form for product, viewing the faciend as operator.
§ 4. Condition of associativity found in commutativity of matrices.
§ 5. Converse proposition and its meaning.
Part 2. The Commutativity of Matrices.
§ 1. Canonical form of any matrix.
§ 2. Form of product of $\phi\psi$ and $\psi\phi$.
§ 3. Form of ϕ when $\phi\psi = \psi\phi$, deduced from square array.
§ 4. Form of ϕ when $\phi\psi = \psi\phi$, deduced from matrix properties.

§ 5. The group of all matrices commutative with a given matrix and some subgroups.
§ 6. Associative units.
§ 7. Equations fulfilled by the numbers of an associative algebra.
Part 3. Linear Associative Numbers.
§ 1. The forms of numbers and their characteristic equations.
§ 2. The units that must belong to the algebra from the form of its equation.
§ 3. Non-quaternionic, quaternionic, and higher systems.
§ 4. Matrical systems.
§ 5. MOLIEN's, SCHEFFERS', and PEIRCE's work.
§ 6. Relations connecting the numbers of an algebra.
§ 7. Table of some of SCHEFFERS' types in associative units.
§ 8. Table of some of PEIRCE's types in associative units.
§ 9. Bases of classification.

Part 1. *The matrix form of a linear algebra.*

§ 1. Let any n independent numbers of an algebra (associative or otherwise) be represented by

$$\rho_1, \rho_2, \rho_3, \cdots, \rho_n,$$

so that the general expression for any number in this algebra is

$$\rho = x_1\rho_1 + x_2\rho_2 + \cdots + x_n\rho_n.$$

Any other number would be

$$\sigma = y_1\rho_1 + y_2\rho_2 + \cdots + y_n\rho_n.$$

The "product" of ρ into σ is defined by the equation

$$\rho\sigma = \Sigma x_i y_j c_{ijk}\rho_k \qquad (i, j, k = 1, 2, \cdots, n),$$

wherein the c_{ijk} are coefficients (scalar) which belong to the algebra in question.

The algebra is *linear* since each set of coefficients enters homogeneously to the first degree.

§ 2. Let us now annex one more unit ρ_0, an ideal unit which enters only to enable us to build the operators which follow. Let

$$\lambda_{ij} \qquad (i, j = 0, 1, \cdots, n),$$

be a matrix operator subject to the definitions

$$\begin{aligned}
\lambda_{i0}\cdot\rho_0 &= \rho_i, \\
\lambda_{i0}\cdot\rho &= 0, \\
\lambda_{ij}\cdot\rho &= x_j\cdot\rho_i, \\
\lambda_{ij}\cdot\lambda_{kl} &= 0 \quad \text{if} \quad j \neq k, \\
&= \lambda_{il} \quad \text{if} \quad j = k.
\end{aligned}$$

Then we may represent the *product* defined above by the matrix form

$$\rho\sigma = \phi_\rho \cdot \sigma$$

wherein

$$\begin{aligned}
\phi_\rho \equiv\ & x_1\lambda_{10} + \lambda_{11}(c_{111}x_1 + c_{211}x_2 + \cdots + c_{n11}x_n) + \lambda_{12}(c_{121}x_1 + \cdots + c_{n21}x_n) \\
& \qquad + \cdots + \lambda_{1n}(c_{1n1}x_1 + \cdots + c_{nn1}x_n) \\
& + x_2\lambda_{20} + \lambda_{21}(c_{112}x_1 + c_{212}x_2 + \cdots + c_{n12}x_n) + \lambda_{22}(c_{122}x_1 + \cdots + c_{n22}x_n) \\
& \qquad + \cdots + \lambda_{2n}(c_{1n2}x_1 + \cdots + c_{nn2}x_n) \\
& + x_3\lambda_{30} + \lambda_{31}(c_{113}x_1 + c_{213}x_2 + \cdots + c_{n13}x_n) + \lambda_{32}(c_{123}x_1 + \cdots + c_{n23}x_n) \\
& \qquad + \cdots + \lambda_{3n}(c_{1n3}x_1 + \cdots + c_{nn3}x_n) \\
& + \ .\ .\ .\

Hence $\phi_i\phi_j$ has the same effect as $\rho_i \cdot \rho_j$. It is to be noticed that

$$\rho_i \cdot \rho_j = \Sigma c_{ijs}\rho_s .$$

Hence

$$\phi_{(\rho_i \cdot \rho_j)} = \Sigma c_{ijs}\lambda_{s0} + \Sigma c_{ijr}c_{rts}\lambda_{st} .$$

In general therefore

$$\phi_{(\rho_i \cdot \rho_j)} \neq \phi_{\rho_i} \cdot \phi_{\rho_j} ,$$

which means that generally the combinations are not associative. It is evident however that a treatment of all combinations, whether associative or not, can be based on these matrix forms.

§ 3. The equation

$$\rho\sigma = \Sigma x_i y_j c_{ijk}\rho_k$$

may also be looked upon as equivalent to the equation

$$\psi_\sigma \cdot \rho = \Sigma x_i y_j c_{ijk}\rho_k ,$$

wherein

$$\psi_\sigma \equiv \Sigma (y_j\lambda_{j0} + y_j c_{ijk} \cdot \lambda_{ki}) .$$

This matrix is analogous to ϕ_ρ. The use of ϕ_ρ is equivalent to conceiving the facient ρ as operating on the faciend σ to produce the result. The use of ψ_σ is equivalent to conceiving the faciend σ as operating on the facient ρ to produce the result.

§ 4. If the numbers are *associative* we must have for any three of them

$$\rho \cdot (\sigma \cdot \tau) \equiv (\rho \cdot \sigma) \cdot \tau ,$$

which gives the matrix form

$$\phi_\rho \cdot \psi_\tau \sigma \equiv \psi_\tau \cdot \phi_\rho \sigma .$$

As this is true for any three numbers, we have the matrix equation

$$\phi_\rho \psi_\tau \equiv \psi_\tau \phi_\rho ,$$

or simply

$$\phi\psi = \psi\phi .$$

This equation defines associativity completely and upon it is to be based the analysis of associative algebras. It includes the theorem of Poincaré* that *two projective groups are connected with every system of complex numbers;* the theorem of STUDY† that *these groups are simply transitive;* the converse

* *Sur les nombres complexes*, Comptes Rendus, t. 99 (1884), pp. 740–742.

† *Complexe Zahlen und Transformationsgruppen*, Leipziger Berichte, 1889, S. 202; *Ueber Systeme Complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen*, Monatsheft f. Math. u. Phys. 1. Jahrg. (1890), S. 332.

of these theorems; and the theorem that these two groups are *commutative.*†
These theorems are *included* in the above equation since the entire set of matrices ϕ or the set ψ may not at all include the matrix 1. The progress of the development of the theory of hyper-complex numbers by means of the group-theory of LIE may be found traced in STUDY's *Ältere und Neuere Untersuchungen über Systeme complexer Zahlen.*‡

§ 5. It is clear that a set of independent ϕ's satisfying the equation

$$\phi\psi = \psi\phi,$$

where ψ is a given matrix, can easily be far more numerous than the order of any field to which ψ may belong. This means that if ψ is a matrix corresponding to some faciend, there are several algebras in which ψ might exist. In fact every set of n independent *matrices* of the set satisfying the equation above, if their products are linearly expressible in terms of these n, gives a linear algebra of n units.

We need now to see what the commutativity of matrices means.

Part 2. *The commutativity of matrices.*

§ 1. Any matrix ψ operating on a field of vectors defined by $\rho_1, \rho_2, \cdots, \rho_n$, may be so expressed, by linear changes in the defining vectors of the field, that if the matrix is written in square form, all coefficients vanish save certain ones on the main diagonal and certain others on the next diagonal to the left. Thus if the equation of lowest degree that ψ satisfies is

$$(\psi - g_1)^{\mu_1}(\psi - g_2)^{\mu_2}\cdots(\psi - g_p)^{\mu_p} \equiv 0,$$

where $g_1 \cdots g_p$ are the p *distinct latent roots*, and where

$$\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_p \equiv n,$$

then there is a region I_{11} of order μ_1', $\mu_1' \geqq \mu_1$, for any vector of which we have the equation

$$(\psi - g_1)^{\mu_1} \cdot I_{11} = 0.$$

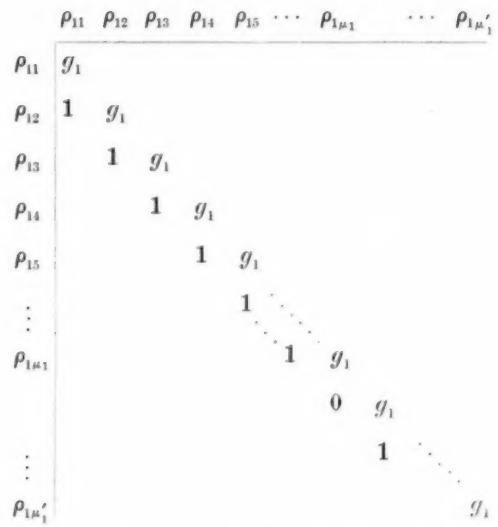
For no vector with a component outside I_{11} is this equation true, for any power of $(\psi - g_1)$. If we choose the vectors defining this region I_{11} properly, the square array of ψ will start as shown in the diagram

---

*Cf. LIE-ENGEL: *Theorie der Transformationsgruppe*, vol. I, S. 367–429.

†Cf. LIE-SCHEFFERS: *Continuierliche Gruppen*, Kapitel 21.

‡ *Mathematical Papers Read at the International Mathematical Congress Held in Chicago, 1893*, pp. 367–381.



The number of *consecutive* 1's in the second diagonal being $\mu_1 - 1$ at least once and nowhere more than $\mu_1 - 1$. For the other roots there are similar regions, and the whole field may be divided up into mutually exclusive regions corresponding to the distinct roots of ψ. The entire diagonal of the array of ψ will then consist of the roots $g_1, g_2 \cdots g_p$ repeated $\mu'_1, \mu'_2 \cdots \mu'_p$ times; the second diagonal will consist of a series of 1's whose number in the consecutive regions will be, in each region of order μ'_i at least $\mu_i - 1$ once, and not more than $\mu_i - 1$ anywhere in that region.

Let λ_{ii0} be the matrix (linear vector operator) which converts

$$x_1\rho_{i1} + x_2\rho_{i2} + \cdots + x_{\mu'_i}\rho_{i\mu'_i}$$

into itself, and annuls all vectors whose components have the form $x_j\rho_{kj}, (i \neq k)$; and further, let ϑ_{iii} convert

$$x_1\rho_{i1} + x_2\rho_{i2} + \cdots + x_{\mu'_i}\rho_{i\mu'_i}$$

into

$$x_1\rho_{i2} + x_2\rho_{i3} + \cdots + x_{\mu'_i-1}\rho_{i\mu'_i}.$$

Then we may write

$$\psi = g_1\lambda_{110} + g_2\lambda_{220} + \cdots + g_p\lambda_{pp0} + \vartheta_{111} + \vartheta_{221} + \cdots + \vartheta_{pp1}.$$

§ 2. Any other matrix ϕ operating in the same field must be representable in the form

$$\phi = \Sigma a_{ij,rs}\epsilon_{ij,rs}$$

where $\epsilon_{ij.rs}$ is a matrix which annuls every vector but ρ_{rs} and converts ρ_{rs} into ρ_{ij}. In square form this means that the constituent of ϕ in the row ij and column rs, is $a_{ij.rs}$. The rows following the vectors $\rho_{i1}, \rho_{i2}, \cdots, \rho_{i\mu_i}$ form a horizontal strip "belonging" to the region i. Likewise the columns under these vectors, form a vertical strip "belonging" to the region i.

If now we form the equation

$$\phi\psi = \psi\phi,$$

from these more expanded forms, we see

(1) $\psi\phi$ has every row in each region multiplied by the corresponding root g, yielding a matrix ϕ', plus a second matrix ϕ'' which is derived from ϕ by dropping each row of constituents in each region one line lower, the first row of the region being filled with zeros, the last row of the region disappearing entirely.

(2) $\phi\psi$ has every column in each region multiplied by the corresponding root g, yielding a matrix ϕ_1, plus a second matrix ϕ_2, which is derived from ϕ by moving each vertical line in a region to the left one place, the line at the right of each region being filled with zeros, and the line at the left of each region disappearing altogether.

§ 3. Suppose now that

$$\phi\psi = \psi\phi.$$

Let us consider first the rectangular block on each side of this equation in the square array of the product and lying in the strip i horizontally and j vertically, $i \neq j$. The matrices ϕ', ϕ'', ϕ_1, ϕ_2 appear thus:

	ρ_{j1}	ρ_{j2}	ρ_{j3}	·
ρ_{i1}	$g_i a_{i1/1}$,	$g_i a_{i1/2}$,	$g_i a_{i1/3}$,	·
ρ_{i2}	$g_i a_{i2/1}$,	$g_i a_{i2/2}$,	$g_i a_{i2/3}$,	·
ρ_{i3}	$g_i a_{i3/1}$,	$g_i a_{i3/2}$,	$g_i a_{i3/3}$,	·
·	·	·	·	·
·	·	·	·	·

$+$

	ρ_{j1}	ρ_{j2}	ρ_{j3}	·
ρ_{i1}	0	0	0	·
ρ_{i2}	$a_{i1/1}$,	$a_{i1/2}$,	$a_{i1/3}$,	·
ρ_{i3}	$a_{i2/1}$,	$a_{i2/2}$,	$a_{i2/3}$,	·
·	·	·	·	·
·	·	·	·	·

	ρ_{j1}	ρ_{j2}	ρ_{j3}	·
ρ_{i1}	$g_j a_{i1/1}$,	$g_j a_{i1/2}$,	$g_j a_{i1/3}$,	·
ρ_{i2}	$g_j a_{i2/1}$,	$g_j a_{i2/2}$,	$g_j a_{i2/3}$,	·
ρ_{i3}	$g_j a_{i3/1}$,	$g_j a_{i3/2}$,	$g_j a_{i3/3}$,	·
·	·	·	·	·
·	·	·	·	·

$+$

	ρ_{j1}	ρ_{j2}	ρ_{j3}	·
ρ_{i1}	$a_{i1/2}$,	$a_{i1/3}$,	$a_{i1/4}$,	·
ρ_{i2}	$a_{i2/2}$,	$a_{i2/3}$,	$a_{i2/4}$,	·
ρ_{i3}	$a_{i3/2}$,	$a_{i3/3}$,	$a_{i3/4}$,	·
·	·	·	·	·
·	·	·	·	·

For the equality to hold we must have then

(1)
$$g_i a_{ij1} = g_j a_{ij1} + a_{ij2},$$
$$g_i a_{ij2} = g_j a_{ij2} + a_{ij3},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$g_i a_{ij\mu_j'} = g_j a_{ij\mu_j'},$$

therefore

$$a_{ij2} = (g_i - g_j) a_{ij1},$$
$$a_{ij3} = (g_i - g_j)^2 a_{ij1},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{ij\mu_j'} = (g_i - g_j)^{\mu_j'-1} a_{ij1},$$
$$0 = (g_i - g_j)^{\mu_j'} a_{ij1}.$$

By hypothesis
$$g_i \neq g_j.$$

Hence
$$0 = a_{ij1} = a_{ij2} = a_{ij3} = \cdots = a_{ij\mu_j'}.$$

(2) Each line yields a similar set of equations, hence all a's of the form a_{ir_j} $(i \neq j)$ must be zero.

This means that ϕ reduces to a sum of matrices each of which acts only in one of the regions of ψ mentioned above.

(3) We may suppose then that ψ consists of one such region only, and that

$$g_1 = g_2 = g_3 = \cdots = g_p.$$

The "regions" may now be taken to be those produced by the 1's in the second diagonal and the 0's that separate them, each set of consecutive 1's lying in a separate region. The equation of the matrix ψ (a matrix still of order n) is

$$(\psi - g)^{\mu} = 0 \qquad (\mu \leqq n),$$

the "widths" of the regions being $\mu_1, \mu_2, \cdots, \mu_p$, where $\mu_1 + \mu_2 + \cdots + \mu_p = n$. Since we may take $g = 1$, with no loss of generality, we may write

$$\psi = \lambda_{110} + \lambda_{220} + \lambda_{330} + \cdots + \lambda_{pp0} + c_1\lambda_{111} + c_2\lambda_{221} + c_3\lambda_{331} + \cdots + c_p\lambda_{pp1},$$

wherein

$$c_i = 0 \quad \text{if} \quad \mu_i = 1,$$
$$c_i = 1 \quad \text{if} \quad \mu_i > 1.$$

In this case the matrices above called ϕ' and ϕ_1 are identical and may be omitted. Hence we have to consider the equality:

$$\begin{array}{c|cccccc} & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdots & \rho_{j\mu_j} \\ \hline \rho_{i1} & 0 & 0 & 0 & \cdots & 0 \\ \rho_{i2} & a_{i1\,j1} & a_{i1\,j2} & a_{i1\,j3} & \cdots & a_{i1\,j\mu_j} \\ \rho_{i3} & a_{i2\,j1} & a_{i2\,j2} & a_{i2\,j3} & \cdots & a_{i2\,j\mu_j} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{i\mu_i} & a_{i\mu_i-1\,j1} & a_{i\mu_i-1\,j2} & a_{i\mu_i-1\,j3} & \cdots & a_{i\mu_i-1\,j\mu_j} \end{array} = \begin{array}{c|cccccc} & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdots & \rho_{j\mu_j} \\ \hline \rho_{i1} & a_{i1\,j2} & a_{i1\,j3} & a_{i1\,j4} & \cdots & 0 \\ \rho_{i2} & a_{i2\,j2} & a_{i2\,j3} & a_{i2\,j4} & \cdots & 0 \\ \rho_{i3} & a_{i3\,j2} & a_{i3\,j3} & a_{i3\,j4} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{i\mu_i} & a_{i\mu_i\,j2} & a_{i\mu_i\,j3} & a_{i\mu_i\,j4} & \cdots & 0. \end{array}$$

It is plain that

$$0 = a_{i1j2} = a_{i1j3} = a_{i1j4} = \cdots = a_{i1j\mu_j} = a_{i2j\mu_j} = \cdots = a_{i\mu_i-1\,j\mu_j}.$$

That is, all the coefficients vanish from the upper row and the right hand line of the block in the strips i and j, except the one in the upper left hand corner and the one in the lower right hand corner.

But we have also

$$a_{irjs} = a_{ir+1\,js+1} \quad (r=1, \cdots, \mu_i-1\,;\ s=1, \cdots, \mu_j-1).$$

This means that all coefficients in the block on a diagonal parallel to the main diagonal are equal. Hence only those coefficients do not vanish which may be formed by filling the diagonals successively with coefficients, all coefficients in one diagonal being equal, and all diagonals which do not run from the left hand line to the lowest line and *vice versa* being zero. For example we may have blocks as follows for values of μ_i, μ_j:

$\mu_i=4,\ \mu_j=3.$

$$\begin{array}{ccc} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_1 & 0 \\ a_3 & a_2 & a_1 \end{array}$$

$\mu_i=4,\ \mu_j=4.$

$$\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_4 & a_3 & a_2 & a_1 \end{array}$$

$\mu_i=3,\ \mu_j=4.$

$$\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \end{array}$$

Let the matrix operator λ_{ijk} convert

$$y_{k+1}\rho_{j1} + y_{k+2}\rho_{j2} + y_{k+3}\rho_{j3} + \cdots y_{k+r}\rho_{jr}$$

into

$$y_{k+1}\rho_{ik+1} + y_{k+2}\rho_{ik+2} + y_{k+3}\rho_{ik+3} + \cdots + y_{k+r}\rho_{ik+r},$$

and be subject to the conditions

$$r \leqq \mu_j, \qquad k + r \leqq \mu_i,$$

or

$$k \geqq \mu_i - \mu_j \quad \text{if} \quad \mu_i > \mu_j,$$

$$k \geqq \mu_i - 1 \quad \text{if} \quad \mu_j \leqq \mu_i.$$

Then the block used above becomes (changing the a's to conform to the λ notation):

$$a_{ij0}\lambda_{ij0} + a_{ij1}\lambda_{ij1} + \cdots + a_{ij\mu_i-1}\lambda_{ij\mu_i-1} \qquad (\mu_i = \mu_j),$$

$$a_{ij0}\lambda_{ij0} + a_{ij1}\lambda_{ij1} + \cdots + a_{ij\mu_i-1}\lambda_{ij\mu_i-1} \qquad (\mu_i < \mu_j),$$

$$a_{ij\mu_i-\mu_j+1}\lambda_{ij\mu_i-\mu_j+1} + a_{ij\mu_i-\mu_j+2}\lambda_{ij\mu_i-\mu_j+2} + \cdots + a_{ij\mu_i-1}\lambda_{ij\mu_i-1} \qquad (\mu_i > \mu_j).$$

This is obviously the sole condition that

$$\phi\psi = \psi\phi.$$

As no restriction was placed upon i and j these results apply to every block formed by the crossing of any horizontal strip and any vertical strip of ϕ.

To illustrate, we may determine the matrix form (general) commutative with

$$\psi = \lambda_{110} + \lambda_{111} + \lambda_{220} + \lambda_{221} + \lambda_{330} + \lambda_{331} + \lambda_{440} + \lambda_{441} + \lambda_{550} + \lambda_{551},$$

whence

$$\mu_1 = 4, \quad \mu_2 = 4, \quad \mu_3 = 3, \quad \mu_4 = 2, \quad \mu_5 = 2.$$

The square array for ϕ would appear thus:

	ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{21}	ρ_{22}	ρ_{23}	ρ_{24}	ρ_{31}	ρ_{32}	ρ_{33}	ρ_{41}	ρ_{42}	ρ_{51}	ρ_{52}
ρ_{11}	a_{110}				a_{120}										
ρ_{12}	a_{111}	a_{110}			a_{121}	a_{120}			a_{131}						
ρ_{13}	a_{112}	a_{111}	a_{110}		a_{122}	a_{121}	a_{120}		a_{132}	a_{131}		a_{142}		a_{152}	
ρ_{14}	a_{113}	a_{112}	a_{111}	a_{110}	a_{123}	a_{122}	a_{121}	a_{120}	a_{133}	a_{132}	a_{131}	a_{143}	a_{142}	a_{153}	a_{152}
ρ_{21}	a_{210}				a_{220}										
ρ_{22}	a_{211}	a_{210}			a_{221}	a_{220}			a_{231}						
ρ_{23}	a_{212}	a_{211}	a_{210}		a_{222}	a_{221}	a_{220}		a_{232}	a_{231}		a_{242}		a_{252}	
ρ_{24}	a_{213}	a_{212}	a_{211}	a_{210}	a_{223}	a_{222}	a_{221}	a_{220}	a_{233}	a_{232}	a_{231}	a_{243}	a_{242}	a_{253}	a_{252}
ρ_{31}	a_{310}				a_{320}				a_{330}						
ρ_{32}	a_{311}	a_{310}			a_{321}	a_{320}			a_{331}	a_{330}		a_{341}		a_{351}	
ρ_{33}	a_{312}	a_{311}	a_{310}	0	a_{322}	a_{321}	a_{320}	0	a_{332}	a_{331}	a_{330}	a_{342}	a_{341}	a_{352}	a_{351}
ρ_{41}	a_{410}				a_{420}				a_{430}			a_{440}		a_{450}	
ρ_{42}	a_{411}	a_{410}	0	0	a_{421}	a_{420}	0	0	a_{431}	a_{430}	0	a_{441}	a_{440}	a_{451}	a_{450}
ρ_{51}	a_{510}				a_{520}				a_{530}			a_{540}		a_{550}	
ρ_{52}	a_{511}	a_{510}	0	0	a_{521}	a_{520}	0	0	a_{531}	a_{530}	0	a_{541}	a_{540}	a_{551}	a_{550}

That is

$$\begin{aligned}
\phi = {} & a_{110}\lambda_{110} + a_{120}\lambda_{120} \\
& + a_{111}\lambda_{111} + a_{121}\lambda_{121} + a_{131}\lambda_{131} \\
& + a_{112}\lambda_{112} + a_{122}\lambda_{122} + a_{132}\lambda_{132} + a_{142}\lambda_{142} + a_{152}\lambda_{152} \\
& + a_{113}\lambda_{113} + a_{123}\lambda_{123} + a_{133}\lambda_{133} + a_{143}\lambda_{143} + a_{153}\lambda_{153} \\
& + a_{210}\lambda_{210} + a_{220}\lambda_{220} \\
& + a_{211}\lambda_{211} + a_{221}\lambda_{221} + a_{231}\lambda_{231} \\
& + a_{212}\lambda_{212} + a_{222}\lambda_{222} + a_{232}\lambda_{232} + a_{242}\lambda_{242} + a_{252}\lambda_{252} \\
& + a_{213}\lambda_{213} + a_{223}\lambda_{223} + a_{233}\lambda_{233} + a_{243}\lambda_{243} + a_{253}\lambda_{253} \\
& + a_{310}\lambda_{310} + a_{320}\lambda_{320} + a_{330}\lambda_{330} \\
& + a_{311}\lambda_{311} + a_{321}\lambda_{321} + a_{331}\lambda_{331} + a_{341}\lambda_{341} + a_{351}\lambda_{351} \\
& + a_{312}\lambda_{312} + a_{322}\lambda_{322} + a_{332}\lambda_{332} + a_{342}\lambda_{342} + a_{352}\lambda_{352} \\
& + a_{410}\lambda_{410} + a_{420}\lambda_{420} + a_{430}\lambda_{430} + a_{440}\lambda_{440} + a_{450}\lambda_{450} \\
& + a_{411}\lambda_{411} + a_{421}\lambda_{421} + a_{431}\lambda_{431} + a_{441}\lambda_{441} + a_{451}\lambda_{451} \\
& + a_{510}\lambda_{510} + a_{520}\lambda_{520} + a_{530}\lambda_{530} + a_{540}\lambda_{540} + a_{550}\lambda_{550} \\
& + a_{511}\lambda_{511} + a_{521}\lambda_{521} + a_{531}\lambda_{531} + a_{541}\lambda_{541} + a_{551}\lambda_{551}.
\end{aligned}$$

Generally we shall write the a's only, or even occasionally omit both a and λ, writing the subscripts only. Thus

$$\begin{aligned}
\phi = {} & 110 \quad 120 \\
& 111 \quad 121 \quad 131 \\
& 112 \quad 122 \quad 132 \quad 142 \quad 152 \\
& 113 \quad 123 \quad 133 \quad 143 \quad 153 \\
& \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

With a further object in view we may write this matrix also in the form

	ρ_{11}	ρ_{21}	ρ_{31}	ρ_{41}	ρ_{51}	ρ_{12}	ρ_{22}	ρ_{32}	ρ_{42}	ρ_{52}	ρ_{13}	ρ_{23}	ρ_{33}	ρ_{14}	ρ_{24}
ρ_{11}	a_{110}	a_{120}													
ρ_{21}	a_{210}	a_{220}													
ρ_{31}	a_{310}	a_{320}	a_{330}												
ρ_{41}	a_{410}	a_{420}	a_{430}	a_{440}	a_{450}										
ρ_{51}	a_{510}	a_{520}	a_{530}	a_{540}	a_{550}										
ρ_{12}	a_{111}	a_{121}	a_{131}			a_{110}	a_{120}								
ρ_{22}	a_{211}	a_{221}	a_{231}			a_{210}	a_{220}								
ρ_{32}	a_{311}	a_{321}	a_{331}	a_{341}	a_{351}	a_{310}	a_{320}	a_{330}							
ρ_{42}	a_{411}	a_{421}	a_{431}	a_{441}	a_{451}	a_{410}	a_{420}	a_{430}	a_{440}	a_{450}					
ρ_{52}	a_{511}	a_{521}	a_{531}	a_{541}	a_{551}	a_{510}	a_{520}	a_{530}	a_{540}	a_{550}					
ρ_{13}	a_{112}	a_{122}	a_{132}	a_{142}	a_{152}	a_{111}	a_{121}	a_{131}			a_{110}	a_{120}			
ρ_{23}	a_{212}	a_{222}	a_{232}	a_{242}	a_{252}	a_{211}	a_{221}	a_{231}			a_{210}	a_{220}			
ρ_{33}	a_{312}	a_{322}	a_{332}	a_{342}	a_{352}	a_{311}	a_{321}	a_{331}	a_{341}	a_{351}	a_{310}	a_{320}	a_{330}		
ρ_{14}	a_{113}	a_{123}	a_{133}	a_{143}	a_{153}	a_{112}	a_{122}	a_{132}	a_{242}	a_{152}	a_{111}	a_{121}	a_{131}	a_{110}	a_{120}
ρ_{24}	a_{213}	a_{223}	a_{233}	a_{243}	a_{253}	a_{212}	a_{222}	a_{232}	a_{242}	a_{252}	a_{211}	a_{221}	a_{231}	a_{210}	a_{220}

That the general form may always be so arranged is easily seen, for the coefficient in any arrangement opposite ρ_{ir} and under ρ_{js} is

$$\begin{aligned} &a_{ij\,r-s} && \text{if} && r \geqq s, \\ &0 && \text{if} && r < s, \\ &0 && \text{if} && r - s < \mu_i - \mu_j. \end{aligned}$$

In this arrangement no block to the right of the main diagonal has any coefficient other than zero. *

This form we shall use frequently, writing only the a's or their subscripts, thus

$$\phi = \begin{matrix} a_{110} & a_{120} & \\ a_{210} & a_{220} & \\ a_{310} & a_{320} & a_{330} \end{matrix}$$

* Cf. DICKSON: *Linear Groups*, p. 229 et seq.

$$\begin{array}{ccccc} a_{410} & a_{420} & a_{430} & a_{440} & a_{450} \\ a_{510} & a_{520} & a_{530} & a_{540} & a_{550} \\ a_{111} & a_{121} & a_{131} & & \\ a_{211} & a_{221} & a_{231} & & \\ a_{311} & a_{321} & a_{331} & a_{341} & a_{351} \\ \text{etc.} & & \text{etc.} & & \end{array}$$

This form is of the highest importance, and the λ's entering it will be called its "*frame*." It will be referred to as *the form* of ϕ when it is commutative with ψ.

§ 4. It is profitable to arrive at these results from another starting point, not using the square array.

(1) If ψ is any matrix operating on a field of n dimensions, the field may be divided up into p mutually exclusive regions corresponding to the p distinct roots. Let these regions be

$$I_{11},\ I_{21},\ \cdots,\ I_{p1},$$

of dimensions $\mu_1', \mu_2', \cdots, \mu_p'$. These regions are found by operating on the whole field by the expressions

$$(g_1 - \psi)^{\mu_1}(g_2 - \psi)^{\mu_2} \cdots (g_{i-1} - \psi)^{\mu_{i-1}}(g_{i+1} - \psi)^{\mu_{i+1}} \cdots (g_p - \psi)^{\mu_p},$$

where $i = 1, 2, \cdots, p$ and $\mu_r \equiv \mu_r'$. This expression reduces the whole field to I_{i1}.

(2) We have

$$(\psi - g_i) I_{j1} = I_{j1} \qquad (i \neq j).$$

But

$$(\psi - g_i) I_{i1} = I_{i2},$$

I_{i2} being a region included in I_{i1} and of dimensions at least one less than μ_i'. If $\mu_i \neq \mu_i'$ the dimensionality may be still smaller than $\mu_i' - 1$. Let it be $\mu_1' - \delta_{i1}$.

Likewise

$$(\psi - g_i) I_{i2} = I_{i3},$$

I_{i3} is included in I_{i2} and is of dimensionality $\mu_1' - \delta_{i1} - \delta_{i2}$, where $\delta_{i2} \leqq \delta_{i1}$.

By repetition of this operation we arrive at $I_{i\mu_i}$ for which

$$(\psi - g_i) I_{i\mu_i} = 0.$$

Of course

$$I_{i3} = (\psi - g_i)^2 I_{i1},$$

$$\cdots\cdots\cdots\cdots$$

$$I_{i\mu_i} = (\psi - g_i)^{\mu_i - 1} I_{i1},$$

$$0 = (\psi - g_i)^{\mu_i} I_{i1}.$$

(3) It is to be noticed that if $\delta_{ik+1} < \delta_{ik}$, say $\delta_{ik} = \delta_{1k+1} + h_{ik}$, then there is a region of dimensionality h_{ik} included in I_{ik}, say I_{ikh}, for which

$$(\psi - g_i) I_{ikh} = 0.$$

The region I_{i1} has a subregion I_{i1e_1} for which

$$(\psi - g_i) I_{i1e_1} = 0.$$

This is made up of parts (generally) of the regions $I_{i1}, I_{i2}, \cdots$. Also, I_{i1} has a subregion I_{i1e_2} for which

$$(\psi - g_i)^2 I_{i1e_2} = 0.$$

This includes I_{i1e_1} of course, and so on.

(4) We call I_{i1e_1} the *first invariant region* of ψ for the root g_i; I_{i1e_2} the *second invariant region.* I_{i1} is the μ_1 *invariant region.*

We call I_{i1} the *first projective region* of ψ for the root g_i; I_{i2} the *second projective region*, etc.

For example, suppose I_{i1} is a region of fifteen dimensions, acted upon by $\psi - g_i$ according to the scheme below, $\psi - g_i$ converting each vector into the one below it, and each projective region being defined by all the vectors below the corresponding bar:

					I_{i1}
ρ_{11}	ρ_{21}	ρ_{31}	ρ_{41}	ρ_{51}	I_{i2}
ρ_{12}	ρ_{22}	ρ_{32}	ρ_{42}	ρ_{52}	I_{i3}
ρ_{13}	ρ_{23}	ρ_{33}			I_{i4}.
ρ_{14}	ρ_{24}				

Then the invariant regions are found by dropping each column until its lowest vector is on the bottom line, each invariant region consisting of all vectors below the corresponding bar:

					I_{i1e_4}
ρ_{11}	ρ_{21}				I_{i1e_3}
ρ_{12}	ρ_{22}	ρ_{31}			I_{i1e_2}
ρ_{13}	ρ_{23}	ρ_{32}	ρ_{41}	ρ_{51}	I_{i1e_1}.
ρ_{14}	ρ_{24}	ρ_{33}	ρ_{42}	ρ_{52}	

That is to say

$$(\psi - g_i)\{\rho_{14}, \rho_{24}, \rho_{33}, \rho_{42}, \rho_{52}\} = 0,$$

and $\psi - g_i$ does not reduce *any other vectors* to 0. So

$$(\psi - g_i)^2\{\rho_{14}, \rho_{24} \cdots \rho_{52}, \rho_{13}, \rho_{23} \cdots \rho_{51}\} = 0,$$

$$(\psi - g_i)^3\{\rho_{14} \cdots \rho_{52}, \rho_{13} \cdots \rho_{51}, \rho_{12} \cdots \rho_{31}\} = 0.$$

Also if ρ is any vector of the region

$$(\psi - g_i)^3\rho = I_{i4} = \{\rho_{14}, \rho_{24}\},$$

$$(\psi - g_i)^2\rho = I_{i3} = \{\rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}\},$$

$$(\psi - g_i)\rho = I_{i2} = \{\rho_{12} \cdots \rho_{52}, \rho_{13} \cdots \rho_{33}, \rho_{14}, \rho_{24}\}.$$

(5) Common to the first projective and last invariant regions are $\{\rho_{11}, \rho_{21}\}$; and $(\psi - g_i)$ successively applied converts this region into $\{\rho_{12}, \rho_{22}\}$, $\{\rho_{13}, \rho_{23}\}$, $\{\rho_{14}, \rho_{24}\}$. These regions together constitute the *shear region* of I_{i1}, of degree 4, or in general μ_i. Again, common to the first projective and third invariant regions is $\{\rho_{31}\}$, giving the *shear region* $\{\rho_{31}, \rho_{32}, \rho_{33}\}$. Common to the first projective and the second invariant regions are $\{\rho_{41}, \rho_{51}\}$ giving the *shear* $\{\rho_{41}, \rho_{51}, \rho_{42}, \rho_{52}\}$.

Suppose now we have two matrices satisfying the relation

$$\phi\psi = \psi\phi.$$

Then

$$(\psi - g_i)\phi = \phi(\psi - g_i),$$

and conversely. If also

$$\psi\{R\} = \{R\};$$

then

$$\phi\psi\{R\} = \phi\{R\} = \psi\phi\{R\},$$

hence if $\{R\}$ is an invariant region of ψ, so is $\phi\{R\}$; also, if

$$(\psi - g_i)\{I\} = \{I'\},$$

then

$$(\psi - g_i)\phi\{I\} = \phi\{I'\};$$

and if

$$(\psi - g_i)I = 0,$$

$$(\psi - g_i)\phi I = 0,$$

$$(\psi - g_i)^2 I = 0,$$

$$(\psi - g_i)^2\phi I = 0,$$

etc.,

so that if the successive projective regions of ψ are

$$I_{i1}, \quad I_{i2}, \quad I_{i3}, \quad I_{i4},$$

the same projective regions are represented by

$$\phi I_{i1}, \quad \phi I_{i2}, \quad \phi I_{i3}, \quad \phi I_{i4}.$$

In the example used above,

$$(\psi - g_i)^3 \cdot \rho = I_{i4},$$

therefore

$$(\psi - g_i)^3 \cdot \phi\rho = \phi I_{i4} = I_{i4},$$

hence ϕ leaves I_{i4} unchanged, or at least adds nothing to it. So

$$(\psi - g_i)^2 \cdot \rho = I_{i3},$$

therefore

$$(\psi - g_i)^2 \cdot \phi I_{i3} = I_{i3}.$$

It follows then that ϕ *does not change the contents of any projective region of* ψ, *but only its internal structure.*

Hence ϕ does not project a vector from one region I_{ii} into another I_{j1}, so that the regions of the roots of ϕ lie in the regions of the roots of ψ, and do not overlap them. Therefore as ϕ must have n dimensions, or the entire field for its operation, we may consider it as made up of matrices each acting only on the ground that the submatrices of ψ act upon, viz. I_{i1}, I_{i2}, etc. In other words we may suppose that all the roots of ψ were equal, and that the example above is of a matrix of order 15, with all its roots equal.

Since

$$(\psi - g)\{\rho_{14}, \rho_{24}, \cdots, \rho_{52}\} = 0,$$

and only for this region is this true, then

$$\phi(\psi - g)\{\rho_{14} \cdots \rho_{52}\} = 0 = (\psi - g)\phi\{\rho_{14} \ldots \rho_{52}\}.$$

Therefore

$$\phi\{\rho_{14} \cdots \rho_{52}\} = \{\rho_{14} \cdots \rho_{52}\}.$$

Likewise

$$\phi\{\rho_{14} \cdots \rho_{52}, \rho_{13} \cdots \rho_{51}\} = \{\rho_{14} \cdots \rho_{51}\},$$

$$\phi\{\rho_{14} \cdots \rho_{51}, \rho_{12}, \rho_{22}, \rho_{31}\} = \{\rho_{14} \cdots \rho_{31}\}.$$

That is, ϕ *does not change the invariant regions of* ψ, *as to their content, but only as to their structure.*

Finally, in the particular case given, since ρ_{14}, ρ_{24} are common to both the fourth projective region and the first invariant region, and since neither changes content, therefore

$$\phi(\rho_{14}, \rho_{24}) = (\rho_{14}, \rho_{24}).$$

Likewise,

$$\phi\rho_{33} = (\rho_{14}, \rho_{24}, \rho_{33}),$$

$$\phi(\rho_{42}, \rho_{52}) = (\rho_{14}, \rho_{24}, \rho_{33}, \rho_{42}, \rho_{52}),$$

$$\phi(\rho_{13}, \rho_{23}) = (\rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}),$$

$$\phi\rho_{32} = (\rho_{32}, \rho_{42}, \rho_{52}, \rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}),$$

$$\phi(\rho_{41}, \rho_{51}) = (\rho_{41}, \rho_{51}, \rho_{32} \cdots \rho_{52}, \rho_{13} \cdots \rho_{33}, \rho_{14}, \rho_{24}),$$

$$\phi(\rho_{12}, \rho_{22}) = (\rho_{12} \cdots \rho_{24}),$$

$$\phi \cdot \rho_{31} \quad = (\rho_{31} \cdots \rho_{24}),$$

$$\phi(\rho_{11}, \rho_{21}) = (\rho_{11} \cdots \rho_{24}).$$

If we let

$$\phi\rho_{13} = x_{13}\rho_{13} + x_{23}\rho_{23} + x_{33}\rho_{33} + x_{14}\rho_{14} + x_{24}\rho_{24},$$

then

$$(\psi - g)\phi\rho_{13} = \phi(\psi - g)\rho_{13} = \phi\rho_{14} = x_{13}\rho_{11} + x_{23}\rho_{24}.$$

If also

$$\phi\rho_{23} = y_{13}\rho_{13} + \cdots + y_{24}\rho_{24},$$

then

$$(\psi - g)\phi\rho_{23} = \phi(\psi - g)\rho_{23} = \phi\rho_{24} = y_{13}\rho_{14} + y_{23}\rho_{24}.$$

Hence if we put

$$\phi \cdot \rho_{24} = a_{120}\rho_{14} + a_{220}\rho_{24},$$

$$\phi \cdot \rho_{14} = a_{110}\rho_{14} + a_{210}\rho_{24},$$

$$\phi \cdot \rho_{33} = a_{330}\rho_{33} + a_{131}\rho_{14} + a_{231}\rho_{24},$$

then

$$\phi \cdot \rho_{23} = a_{120}\rho_{13} + a_{220}\rho_{23} + a_{320}\rho_{33} + a_{121}\rho_{14} + a_{221}\rho_{24},$$

$$\phi\rho_{13} = a_{110}\rho_{13} + a_{210}\rho_{23} + a_{310}\rho_{33} + a_{111}\rho_{14} + a_{211}\rho_{24}.$$

A full set of equations which amount to the table already given for ϕ, may easily be deduced.

The first arrangement of the constituents of ϕ, it will be observed, is according to the shear regions, the second according to the projective region.

§ 5. This also gives the entire group of matrices commutative with a given matrix whose roots are all equal. With regard to this group we observe the following:

(1) Let the symbol $S_a^{(b)}$ designate any part of any shear region common to the projective region of order a and the invariant region of order b. If the shear order is μ,

$$a + b = \mu + 1.$$

Then the group leaves invariant the regions made up of $S_a^{(b)}$ and all subregions $S_{a'}^{(b')}$ such that

$$b' \leqq b_1, \qquad a' \geqq a_1.$$

For example in the above let

$$(\rho_{41}, \rho_{51}) = S_1^{(2)},$$

then the group leaves invariant

$$(S_1^{(2)}, S_2^{(2)}, S_4^{(1)}, S_3^{(1)}, S_2^{(1)}),$$

that is,

$$(\rho_{41}, \rho_{51}, \rho_{32}, \rho_{14}, \rho_{24}, \rho_{13}, \rho_{23}, \rho_{33}).$$

If the shears are all of *width* unity, it is obvious that there is for the group a set of invariant regions, of orders one, two, three, $\cdots$ to n, each region including the preceding.* The theorem above is to the effect that each group of matrices commutative with a given matrix whose roots are all equal leaves invariant a system of regions, which in the main are each included in the succeeding. The exceptions arise from the character of the subregions.

For example, ϕ_a above leaves invariant

$$(\rho_{14}, \rho_{24}), \quad (\rho_{33}, \rho_{14}, \rho_{24}), \quad (\rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}), \quad (\rho_{42}, \rho_{52}, \rho_{33}, \rho_{14}, \rho_{24}), \text{ etc.}$$

The fourth does not include the third.

(2) It is obvious that the characteristic equation of any matrix of the group is the product of the determinants along its diagonal, with $-\phi$ inserted after every diagonal coefficient. For the special case above the equation is

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 [a_{330}-\phi]^3 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 \equiv 0.$$

This equation obviously may degenerate for some of the matrices of the group.

Let the determinant factors of the general characteristic equation for any frame be represented by $\Phi_1, \Phi_2, \cdots, \Phi_e$, of multiplicities $m_1, m_2, \cdots, m_e$, and widths $w_1, w_2, \cdots, w_e$. These factors we shall call *shear factors*. The general equation is then

$$\Phi_1^{m_1} \Phi_2^{m_2} \cdots \Phi_e^{m_e} \equiv 0.$$

Thus, in the example above

$$\Phi_1 = \begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix} \qquad (m_1 = 4,\ w_1 = 2),$$

$$\Phi_2 = a_{330} - \phi \qquad (m_2 = 3,\ w_2 = 1),$$

$$\Phi_3 = \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix} \qquad (m_3 = 2,\ w_3 = 2).$$

In any case

$$n = w_1 m_1 + w_2 m_2 + \cdots + w_e m_e.$$

In a degenerate equation we have two cases, either

(a) all the shear factors are present,

(b) some shear factors are absent.

If we consider the shear factors we notice that any one of them as Φ_c reduces a certain region, which we may call the *shear region* c, by the width w_c and

*Cf. LIE-ENGEL: *Theorie der Transformationsgruppen*, vol. I, p. 589, Satz 4.

also projects it into other shear regions (the internal change in the shear region c is immaterial.) If this factor is

$$\Phi_c = \begin{vmatrix} a_{r_c r_c 0} - \phi & a_{r_c r_c+1\,0} & \cdots & a_{r_c r_c+s_c\,0} \\ a_{r_c+1\,r_c 0} & a_{r_c+1 r_c+1 0} - \phi & \cdots & a_{r_c+1 r_c+s_c\,0} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r_c+s_c\,r_c 0} & a_{r_c+s_c\,r_c+1\,0} & \cdots & a_{r_c+s_c\,r_c+s_c\,0} - \phi \end{vmatrix},$$

since it has the effect pointed out, it cannot contain in its expanded form in terms of the λ's any of the block:

$$\begin{matrix} \lambda_{r_{c_1} r_{c_1} 0} & \lambda_{r_{c_1} r_{c_1}+1, 0} & \cdots & \lambda_{r_{c_1} r_{c_1}+s_{c_1} 0} \\ \lambda_{r_{c_1}+1, r_{c_1} 0} & \lambda_{r_{c_1}+1, r_{c_1}+1, 0} & \cdots & \lambda_{r_{c_1}+1, r_{c_1}+s_{c_1} 0} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{r_{c_1}+s_{c_1} r_{c_1} 0} & & \cdots & \lambda_{r_{c_1}+s_{c_1} r_{c_1}+s_{c_1} 0}. \end{matrix}$$

The presence of any one of these would prevent the reduction that must take place.

Thus in the above, Φ_1 cannot contain $\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220}$; Φ_2 cannot contain λ_{330}; Φ_3 cannot contain $\lambda_{440}, \lambda_{450}, \lambda_{540}, \lambda_{550}$.

It is obvious that $\Phi_1 \Phi_2$ operating on $(\rho_{11}, \rho_{21}, \rho_{31})$ projects this region outside itself, hence $\Phi_1 \Phi_2$ cannot contain $\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220}, \lambda_{310}, \lambda_{320}, \lambda_{330}$. Operating on $(\rho_{33}, \rho_{14}, \rho_{24})$ it annuls the region completely, hence it cannot contain λ_{131}, λ_{231}. For a similar reason, $\Phi_1 \Phi_2 \Phi_3$ cannot contain $\lambda_{110}, \cdots, \lambda_{210}, \cdots, \lambda_{310}, \cdots, \lambda_{410}, \cdots, \lambda_{510}, \cdots, \lambda_{550}$. Likewise $\Phi_1^2 \Phi_2$ cannot contain $\lambda_{110}, \cdots, \lambda_{220}, \lambda_{310}, \cdots, \lambda_{330}, \lambda_{111}, \lambda_{121}, \lambda_{131}, \lambda_{211}, \lambda_{221}, \lambda_{231}$. Further products and the λ's that can be determined to be missing from their expression can easily be traced from the diagram in Part 2, § 3 (3). The generalization for any frame is obvious.

(3) If we have an equation of order lower than the most general one belonging to the frame, this equation will include *every* shear factor to some power, or only *part* of the shear factors. In the first case let it be

$$\Phi_1^{m_1'} \Phi_2^{m_2'} \Phi_3^{m_3'} \cdots \Phi_e^{m_e'} \equiv 0 \qquad (m_i' \gtrless m_i,\ m_1 \gtrless m_2 \gtrless m_3 \cdots \gtrless m_e)$$

The coefficients in the determinantal form of the shear factors we suppose to be arbitrary (the only case interesting us here). If we write the expression

$$\Psi = \Phi_1^{m_1'} \Phi_2^{m_2'} \cdots \Phi_e^{m_e'-1},$$

this expression, since it does not vanish by hypothesis, *must* contain some of the terms, and can contain only the following:

$$\left.\begin{matrix} \lambda_{r_e+t,\, r_e+u,\, 0} \\ \lambda_{r_e+t,\, r_e+u,\, 1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_{r_e+t,\, r_e+u,\, m_e-1} \end{matrix}\right\} \qquad (t,\, u = 0,\, 1 \cdots s_e).$$

In Ψ let all coefficients become zero, which have the form

$$a_{r_e+t,\, r_e+u,\, 0};$$

let Ψ' represent this value of Ψ; then since by hypothesis Ψ' cannot vanish until multiplied by the corresponding value of Φ_e, or $(-\phi)^{m_e}$, it must follow that Ψ' contains terms of the form

$$\lambda_{r_e+t,\, r_e+u,\, i} \qquad (0 < i < m_e).$$

If we let the aggregate of terms of the form

$$a_{r_e+t,\, r_e+u,\, 0}\, \lambda_{r_e+t,\, r_e+u,\, 0}$$

be represented by Ψ_0, the remaining ones being Ψ', then $\Psi = \Psi_0 + \Psi'$.

If we raise Ψ to a high enough power, since no power of it can vanish, and since the coefficients of Ψ_0 are arbitrary, and since the powers of Ψ_0 contain the same λ's, it follows that we arrive at a series of non-vanishing expressions or matrices, which may be so added and subtracted as to yield each of the matrices

$$\lambda_{r_e+t,\, r_e+u,\, 0}.$$

These elementary matrices belong then to the subgroup consisting of all matrices having this equation. That being the case, their products into Ψ' must yield matrices of the form

$$\lambda_{r_e+t,\, r_e+u,\, i} \qquad (b < i < m_e)$$

as members of the same subgroup. The lower limit of i, namely b, must be determined from the fact that the lowest of these raised to the m_e' power must vanish; that is

$$\lambda^{m_e'}_{r_e+t,\, r_e+t,\, b+1} = 0,$$

whence

$$m_e'(b+1) > m_e,$$

giving

$$b > \frac{m_e}{m_e'} - 1.$$

Obviously $\lambda_{r_e+t,\, r_e+u,\, k}$ is produced by the product $\lambda_{r_e+t,\, r_e+t,\, k} \cdot \lambda_{r_e+t,\, r_e+u,\, 0}$. Hence $m_e > i > m_e/m_e' - 1$.

These λ's being removed from the general expression of ϕ, we proceed to find those for the other factors.

The subgroup then of all matrices whose equation is of the form

$$\Phi_1^{m_1'}\Phi_2^{m_2'}\cdots\Phi_e^{m_e'} \equiv 0 \qquad (m_i' \leqq m_i),$$

contains all the elementary matrices

$$\lambda_{r_i+t,\, r_i+u,\, 0} \qquad (i=1, 2, \cdots, e;\ t, u=0, 1, \cdots, s_i),$$

$$\lambda_{r_i+t,\, r_i+u,\, k_i} \qquad \left(m_i > k_i > \frac{m_i}{m_i'} - 1\right).$$

Any other matrix of the subgroup must then depend on forms not included in these, and *not determinable from the equation.*

If the equation lacks some of the shear factors, then some of the coefficients in the wider shear factor are equal to those in the narrower which may be considered as similarly placed. In this case the problem is more complicated, though to be handled on these same general lines. For example, in the special case used above, if

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 \equiv 0,$$

Then $a_{330} = a_{110}$ or a_{220}, it is immaterial which. The forms corresponding to Φ_1 become $\lambda_{110}+\lambda_{330}$, λ_{120}, λ_{210}, λ_{220}, λ_{111}, λ_{331}, λ_{121}, λ_{211}, λ_{221}, λ_{112}, λ_{332}, λ_{122}, λ_{212}, λ_{222}, λ_{113}, λ_{123}, λ_{213}, λ_{223}.

§ 6. The most striking feature of the form which any number takes, is that it is expressible linearly in terms of the elementary matrices λ_{rst}. In fact these forms may be considered to be the elementary units, called *associative units*, of a general algebra, which is of necessity linear and associative. They have from this point of view the definitions

$$\lambda_{rst}\lambda_{r's't'} = c\vartheta_{ss'}\lambda_{rs't+t'} \qquad (\mu_r - 1 \leqq t+t' \leqq \mu_r - \mu_{s'}),$$

where $\vartheta_{ss'} = 1$ if $s = r'$, 0 if $s \neq r'$, $c = 1$ if $t+t'$ satisfies the condition, 0 if $t+t'$ does not satisfy the condition.

The group of commutative matrices is then representable by $\Sigma\, a_{rst}\lambda_{rst}$, which is commutative with

$$\lambda_{110} + \lambda_{111} + \lambda_{220} + \lambda_{221} + \cdots + \lambda_{pp0} + \lambda_{pp1}.$$

It is worth noticing that if

$$i \neq j, \quad \mu_i = \mu_j,$$

then

$$\lambda_{ij0}\lambda_{ji0} = \lambda_{ii0},$$

$$\lambda_{ji0}\lambda_{ij0} = \lambda_{jj0}.$$

If

$$\lambda' = \lambda_{ij0}\lambda_{ji0} - \lambda_{ji0}\lambda_{ij0} = \lambda_{ii0} - \lambda_{jj0},$$

then

$$\lambda'\lambda_{ij0} - \lambda_{ij0}\lambda' = 2\lambda_{ij0},$$

$$\lambda'\lambda_{ji0} - \lambda_{ji0}\lambda' = -2\lambda_{ji0}.$$

The four forms $\lambda_{ii0}, \lambda_{jj0}, \lambda_{ij0}, \lambda_{ji0}$ form a quaternion matrix group, i. e. the general forms

$$a_{ii0}\lambda_{ii0} + a_{ij0}\lambda_{ij0} + a_{ji0}\lambda_{ji0} + a_{jj0}\lambda_{jj0}$$

obey the laws of quaternions. In fact these four units are practically the four canonical quaternion units.

So if

$$i \neq j \neq k, \qquad \mu_i = \mu_j = \mu_k,$$

we have the nonion group

$$\begin{matrix} \lambda_{ii0} & \lambda_{ij0} & \lambda_{ik0} \\ \lambda_{ji0} & \lambda_{jj0} & \lambda_{jk0} \\ \lambda_{ki0} & \lambda_{kj0} & \lambda_{kk0}. \end{matrix}$$

For these we have

$$\lambda_{ij0}\lambda_{jk0}\lambda_{ki0} = \lambda_{ii0}, \text{ etc.},$$

If $\omega^3 = 1$, and

$$\lambda' = \lambda_{ij0}\lambda_{jk0}\lambda_{ki0} + \omega\lambda_{jk0}\lambda_{ki0}\lambda_{ij0} + \omega^2\lambda_{ki0}\lambda_{ij0}\lambda_{jk0} = \lambda_{ii0} + \omega\lambda_{jj0} + \omega^2\lambda_{kk0},$$

then

$$\lambda'^2 = \lambda_{ij0}\lambda_{jk0}\lambda_{ki0} + \omega^2\lambda_{jk0}\lambda_{ki0}\lambda_{ij0} + \omega\lambda_{ki0}\lambda_{ij0}\lambda_{jk0} = \lambda_{ii0} + \omega^2\lambda_{jj0} + \omega\lambda_{kk0},$$

$$\lambda'\lambda_{ij0}\lambda_{jk0} + \omega\lambda_{ij0}\lambda'\lambda_{jk0} + \omega^2\lambda_{ij0}\lambda_{jk0}\lambda' = \lambda_{ik0}(1 + \omega^2 + \omega) = 0,$$

$$\lambda'\lambda_{ij0}\lambda_{jk0} + \omega^2\lambda_{ij0}\lambda'\lambda_{jk0} + \omega\lambda_{ij0}\lambda_{jk0}\lambda' = 3\lambda_{ik0}, \text{ etc.}$$

Obviously any matrix may be put into a similar form.

There are other sets of these forms, as for example $\lambda_{110} + \lambda_{220}$, λ_{111}, λ_{221}, λ_{210} λ_{310}, λ_{330}, which involve the compound idempotents $\lambda_{110} + \lambda_{220}$, etc.

§7. The matrix form given yields the equations

$$\phi\cdot\rho_{11} = a_{110}\rho_{11} + a_{210}\rho_{21} + \cdots + a_{113}\rho_{14} + a_{213}\rho_{24},$$

$$\phi\cdot\rho_{12} = a_{120}\rho_{11} + a_{220}\rho_{22} + \cdots + a_{123}\rho_{14} + a_{223}\rho_{24},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi\cdot\rho_{14} = a_{110}\rho_{14} + a_{210}\rho_{24},$$

$$\phi\cdot\rho_{24} = a_{120}\rho_{14} + a_{220}\rho_{24}.$$

Inasmuch as these correspond to matrix equations of the algebra, we must have, if ϕ_{11} corresponds to ρ_{11}, etc.,

$$\phi \cdot \phi_{11} = a_{110}\phi_{11} + a_{210}\phi_{21} + \cdots + a_{113}\phi_{14} + a_{213}\phi_{24},$$

$$\phi \cdot \phi_{12} = a_{120}\phi_{11} + a_{220}\phi_{22} + \cdots + a_{123}\phi_{14} + a_{223}\phi_{24},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi \cdot \phi_{14} = a_{110}\phi_{14} + a_{210}\phi_{24},$$

$$\phi \cdot \phi_{24} = a_{120}\phi_{14} + a_{220}\phi_{24}.$$

From the last two equations

$$\begin{vmatrix} a^{(24)}_{110} - \phi_{24} & a^{(24)}_{210} \\ a^{(24)}_{120} & a^{(24)}_{220} - \phi_{24} \end{vmatrix} \phi_{24} = 0,$$

$$\begin{vmatrix} a^{(14)}_{110} - \phi_{14} & a^{(14)}_{210} \\ a^{(14)}_{120} & a^{(14)}_{220} - \phi_{14} \end{vmatrix} \phi_{14} = 0.$$

We have further

$$\begin{vmatrix} a^{(24)}_{110} - \phi_{24} & a^{(24)}_{210} \\ a^{(24)}_{120} & a^{(24)}_{220} - \phi_{24} \end{vmatrix} \phi_{14} = 0, \qquad \begin{vmatrix} a^{(14)}_{110} - \phi_{14} & a^{(14)}_{210} \\ a^{(14)}_{120} & a^{(14)}_{220} - \phi_{14} \end{vmatrix} \phi_{24} = 0.$$

There are other equations easily written down from the equations above. These are useful in determining the laws of algebras as well as structure. The consequences of these equations are of great importance, but are deferred for further consideration.

Part 3. *Linear associative algebras.*

§ 1. We may now enunciate as the result of the preceding analysis the following proposition:

Any number of any linear associative algebra of n units is of the form

$$\phi = \Sigma \cdot a_{rst}\lambda_{rst} \qquad (r, s = 1, \cdots, p),$$

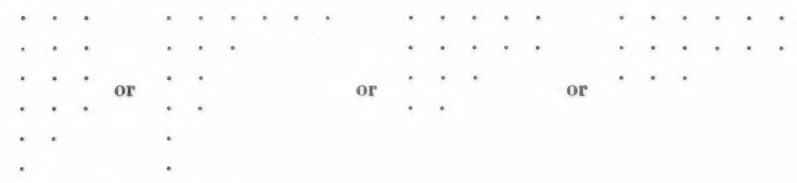
and for each r or s exists a multiplicity number μ_r or μ_s; these numbers satisfy the conditions $\mu_1 + \mu_2 + \cdots + \mu_p = n + 1$; $\mu_r \sim \mu_s \equiv t < \mu_r$.

We have $n + 1$ and not n in this equation just above, because of the ideal unit that we added to the list of units (Part 1, § 2). This ideal unit plays no part in the theory. The form of ϕ may be shown in diagrams similar to those of Part 2, § 3.

The number ϕ satisfies an equation of degree $n + 1$ or less, and the factors of this equation are certain determinants, the shear factors mentioned before. The constituents of these determinants are arbitrary, also the degree of this

equation being less than $n+1$, there are two distinct types of equation: (1) *when the factors belonging to the frame are all represented,* (2) *when some of the factors are not represented.*

If the order is n, we may have in the equation such factors and multiplicities as can occur in the partitions of $n+1$. Thus if $n=14$, we may have among other forms



corresponding to the cases:

$\mu_1=6,\ w_1=1,\quad \mu_2=5,\ w_2=1,\quad \mu_3=4,\ w_3=1;$

$\mu_1=6,\ w_1=1,\quad \mu_2=4,\ w_2=1,\quad \mu_3=2,\ w_3=1,\quad \mu_4=1,\ w_4=1,\quad \mu_5=1,\ w_5=1,\quad \mu_6=1,\ w_6=1;$

$\mu_1=4,\ w_1=2,\quad \mu_2=3,\ w_2=1,\quad \mu_3=2,\ w_3=2;$

$\mu_1=3,\ w_1=3,\quad \mu_2=2,\ w_1=3.$

The general equations of the corresponding frames would be respectively:

$$(a_{110}-\phi)^6(a_{220}-\phi)^5(a_{330}-\phi)^4=0,$$

$$(a_{110}-\phi)^6(a_{220}-\phi)^4(a_{330}-\phi)^2(a_{440}-\phi)(a_{550}-\phi)(a_{660}-\phi)=0,$$

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 [a_{330}-\phi]^3 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 = 0,$$

$$\begin{vmatrix} a_{110}-\phi & a_{120} & a_{130} \\ a_{210} & a_{220}-\phi & a_{230} \\ a_{310} & a_{320} & a_{330}-\phi \end{vmatrix}^3 \begin{vmatrix} a_{440}-\phi & a_{450} & a_{460} \\ a_{540} & a_{550}-\phi & a_{560} \\ a_{640} & a_{650} & a_{660}-\phi \end{vmatrix}^2 = 0.$$

These equations might degenerate for algebras of each class. The characteristic equation, if the degree is less than $n+1$, might thus arise in different ways. For example, if an algebra of the first frame above had the equation

$$(a_{110}-\phi)^6(a_{220}-\phi)^4=0,$$

the units might include $\lambda_{110}+\lambda_{330}$ or $\lambda_{220}+\lambda_{330}$, according to whether $a_{330}\equiv a_{110}$ or $a_{330}\equiv a_{220}$.

The frames that are possible determine the possible associative numbers. There may be for a given frame, obviously, more numbers expressible than the order of the frame, e. g. for the frame

110						110		
210	220					210	220	
111	121	110			or	111	121	
211	221	210	220			211	221	
112	122	111	121	110		112	122	

which is of order *five*, we may have as distinct numbers the *nine*

$$\lambda_{110},\ \lambda_{220},\ \lambda_{210},\ \lambda_{111},\ \lambda_{121},\ \lambda_{211},\ \lambda_{221},\ \lambda_{112},\ \lambda_{122}.$$

We might look upon any algebra built upon this frame as a sub-algebra of this one of order nine. The general equation of the algebra is

$$(a_{110} - \phi)^3 (a_{220} - \phi)^2 = 0.$$

But this is equally the equation of the sub-algebra

$$\lambda_{110},\ \lambda_{111},\ \lambda_{112},\ \lambda_{220},\ \lambda_{221},\ \lambda_{210},\ \lambda_{211}.$$

On the other hand the sub-algebra of five units

$$\lambda_{110} + \lambda_{220},\qquad \lambda_{210} + a\lambda_{121},\qquad \lambda_{111} + \lambda_{221},\qquad \lambda_{211} + a\lambda_{122},\qquad \lambda_{112},$$

has the equation

$$(a_{110} - \phi)^5 = 0,$$

and is isomorphic with $\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{114}$ of a different frame. Evidently the equation of a frame cannot have a higher degree than the frame, and if all species of the general equation are considered we have corresponding species of sub-algebras, which may each consist of many individual algebras. The equation just given might however have arisen from some other general equation, belonging to a different frame.

§ 2. *Every algebra or group of numbers whose characteristic equation is*

$$\Phi_1^{m_1'} \Phi_2^{m_2'} \cdots \Phi_e^{m_e'} = 0,$$

(the factors Φ_i being of width w_i) and belonging to a frame whose general equation is

$$\Phi_1^{m_1} \Phi_2^{m_2} \cdots \Phi_e^{m_e} = 0 \qquad (m_i \leqq m_i'),$$

must contain the associative units

$$\lambda_{r_i+t,\, r_i+u,\, 0} \qquad (i=1, 2, \cdots, e),$$

$$\lambda_{r_i+t,\, r_i+u,\, k_i} \quad \left(t, u=0, 1, \cdots, s_i;\ m_i > k_i > \frac{m_i}{m_i'}-1\right),$$

by Part 2, § 5 (3).

For example, in SCHEFFER's V_8,* with equation

$$(x-x_5e)(x-x_4e)(x-x_3e)=0$$

we have

$$e_5=\lambda_{330}, \quad e_4=\lambda_{220}, \quad e_3=\lambda_{110}, \quad e_2=\lambda_{310}, \quad e_1=\lambda_{320}.$$

The frame is of order 5, general equation

$$(x-x_5e)^2(x-x_4e)^2(x-x_3e)=0.$$

V_9 is from the same frame, but

$$e_5=\lambda_{330}, \quad e_4=\lambda_{220}, \quad e_3=\lambda_{110}, \quad e_2=\lambda_{210}, \quad e_1=\lambda_{320}.$$

V_{10} has the equation $(x-x_4e)(x-x_5e)^2=0$, and this equation may come from either

$$(x-x_4e)^3(x-x_5e)^2=0 \quad \text{or} \quad (x-x_4e)^2(x-x_5e)^3=0.$$

The first corresponds to V_{10}, the second to V_{11}, which has the same equation, giving

V_{10}, $\quad e_5=\lambda_{220}, \quad e_4=\lambda_{110}, \quad e_3=\lambda_{221}, \quad e_2=\lambda_{211}, \quad e_1=\lambda_{210}.$

V_{11}, $\quad e_5=\lambda_{110}, \quad e_4=\lambda_{220}, \quad e_3=\lambda_{112}, \quad e_2=\lambda_{121}, \quad e_1=\lambda_{122}.$

All algebras of this type for the order 5 may be arrived at by considering the general equations of order 5 which can reduce without losing any shear factor. These are evidently, $(x-x_0)^5$, $(x-x_0)^4(x-x_1)$, $(x-x_0)^3(x-x_1)(x-x_2)$, $(x-x_0)^3(x-x_1)^2$, $(x-x_0)^2(x-x_1)(x-x_2)(x-x_3)$, $(x-x_0)^2(x-x_1)^2(x-x_2)$, giving respectively $(x-x_0)^4$, $(x-x_0)^3$, $(x-x_0)^2$,

$$(x-x_0)^3(x-x_1),\ (x-x_0)^2(x-x_1),\ (x-x_0)(x-x_1),$$

$$(x-x_0)^2(x-x_1)(x-x_2),\ (x-x_0)(x-x_1)(x-x_2),$$

$$(x-x_0)^2(x-x_1)^2,\ (x-x_0)(x-x_1)^2,\ (x-x_0)^3(x-x_1),$$

$$(x-x_0)^2(x-x_1),\ (x-x_0)(x-x_1),$$

$$(x-x_0)(x-x_1)(x-x_2)(x-x_3),$$

$$(x-x_0)^2(x-x_1)(x-x_2),\ (x-x_0)(x-x_1)(x-x_2).$$

* Mathematische Annalen, vol. 39 (1891), S. 356.

The equations which are alike but derived from different equations, would generally belong to algebras with different multiplication tables, which shows again that the characteristic equation alone does not determine the algebra, though it conditions it.

We have to consider next the case in which the shear factors coalesce, giving an equation of the form

$$\Phi_1^{m_1'}\Phi_2^{m_2'}\cdots\Phi_c^{m_c'}=0 \qquad (c<e).$$

The only way in which this can happen is by certain identities between the otherwise arbitrary constituents being added to the general equation, e. g., let the general equation be

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 (a_{330}-\phi)^3 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 = 0,$$

the special equation being

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^3 = 0.$$

In this case the factors Φ_2, Φ_3, have been absorbed into Φ_1, that is, the operation of Φ_1^3 on the field annuls it altogether. This must mean that the operation of Φ_1 is equivalent to the simultaneous operation of $\Phi_1\Phi_2\Phi_3$, or that

$$a_{110}\equiv a_{330}\equiv a_{440},$$
$$a_{120}\equiv a_{450},$$
$$a_{210}\equiv a_{540},$$
$$a_{220}\equiv a_{550}.$$

Therefore there must be involved in ϕ the compound units

$$\lambda_{110}+\lambda_{330}+\lambda_{440},$$
$$\lambda_{220}+\lambda_{550},$$
$$\lambda_{120}+\lambda_{450},$$
$$\lambda_{210}+\lambda_{540},$$

so that

$$\phi = a_{110}(\lambda_{110}+\lambda_{330}+\lambda_{440})+a_{220}(\lambda_{220}+\lambda_{550})+a_{120}(\lambda_{120}+\lambda_{450}) + a_{210}(\lambda_{210}+\lambda_{540})+\cdots.$$

The effect of Φ_1 on the field is to reduce the field by all the vectors which belong to the region $\rho_{11}, \rho_{21}, \rho_{31}, \rho_{41}, \rho_{51}$. In general the coalesced shear factor

will reduce the corresponding shear regions into the second invariant region. A further application will reduce the same into the third invariant region. If the factor Φ_1^2 or $\Phi_1^3 \cdots$ occurs in the equation of the group of algebras under discussion, there must be terms of the form

$$\Sigma a_{rst} \lambda_{rst}$$

or

$$\Sigma a_{rs3} \lambda_{rs3} + \Sigma a_{rst} \lambda_{rst},$$

and so on, according to the character of the coalescence. The further discussion of this case must be reserved for another paper. It is to be observed that a form arising thus may be isomorphic with a simpler form. There is in fact a sort of resemblance between the relation of these associative units to the algebras they represent and the relation of substitutions to abstract groups.

The determination of all forms of units for the equation

$$\Phi_1^{m'_1} = 0$$

which has only one shear factor, is obviously the basal problem of the subject. The determination of these units enables us to determine corresponding units for any equation of the form

$$\Phi_1^{m'_1} \Phi_2^{m'_2} \cdots = 0.$$

These may be called *direct units*. Units of the form

$$\Sigma a_{rst} \lambda_{rst}$$

where r and s belong to different shear regions are called *oblique units*. The square of any such obviously vanishes. The determination for a given frame and given direct units is a very simple matter.

§ 3. The frames which have shears whose width is *unity* only, include SCHEFFERS' *nonquaternionic** algebras. In none of his cases can any diagonal coefficient a_{rr0} vanish, since all his algebras imply a modulus. The η's of this treatment are the forms

$$\lambda_{110}, \ \lambda_{220},$$

or

$$\lambda_{110} + \lambda_{220}, \text{ etc.}$$

Obviously the e's fall into the two classes

$$\Sigma a_{rrt} \lambda_{rrt},$$

$$\Sigma a_{rst} \lambda_{rst} \qquad (r \neq s).$$

The equation is of the form

$$(x - x_1)^{\mu_1} (x - x_2)^{\mu_2} \cdots (x - x_s)^{\mu_s} = 0.$$

*Complexe Zahlensysteme, Mathematische Annalen, vol. 39 (1891), pp. 293–390.

A table of the expression of SCHEFFERS' nonquaternionic forms up to order 5 follows in a later section.

The frames which have shears whose width is *two*, include for every such shear a *quaternionic* system. To such a factor correspond as necessary members of the algebra

$$\lambda_{rr0} \quad \lambda_{rr+1\,0}$$

$$\lambda_{r+1\,r0} \quad \lambda_{r+1\,r+1\,0}$$

and these are essentially quaternionic, in the canonical form of quaternions. The corresponding factor of the equation is

$$\begin{vmatrix} x_1 - x & x_2 \\ x_3 & x_4 - x \end{vmatrix}^m,$$

or equally

$$[x^2 - 2x_0 x + x_0^2 + x_1^2 + x_2^2 + x_3^2]^m.$$

If a system is of order more than four, and its equation contains the factor

$$\begin{vmatrix} x_1 - x & x_2 \\ x_3 & x_4 - x \end{vmatrix}^m$$

it *must* contain the quaternion system and also forms of the type

$$\lambda_{rr1} \quad \lambda_{r,\,r+1,\,1}$$

$$\lambda_{r+1,\,r,\,1} \quad \lambda_{r+1,\,r+1,\,1}$$

etc.

This extends the theory given by SCHEFFERS.*

It is obvious that if the frame of any algebra is of the form

110	120				
210	220				
111	121	110	120		
211	221	210	220		
112	122	111	121	110	120
212	222	211	221	210	220
.	.	.	.	.	.
.	.	.	.	.	.

* Loc. cit., S. 364.

then it contains the units

$$\{\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220}\} = \{Q\},$$

$$\{\lambda_{111}, \lambda_{121}, \lambda_{211}, \lambda_{221}\} = \{Q\}(\lambda_{111} + \lambda_{221}) = (\lambda_{111} + \lambda_{221})\{Q\},$$

$$\{\lambda_{112}, \lambda_{122}, \lambda_{212}, \lambda_{222}\} = \{Q\}(\lambda_{112} + \lambda_{222}) = (\lambda_{112} + \lambda_{222})\{Q\}.$$

If we have the frame below,

110	120				
210	220				
310	320	330	340		
410	420	430	440		
111	121	131	141	110	120
211	221	231	241	210	220

and if also

$$a_{110} = a_{330}, \quad a_{120} = a_{340}, \quad a_{210} = a_{430}, \quad a_{220} = a_{440},$$

then we must have the units

$$\{\lambda_{110} + \lambda_{330}, \lambda_{120} + \lambda_{340}, \lambda_{210} + \lambda_{430}, \lambda_{220} + \lambda_{440}\} = \{Q\},$$

$$\{\lambda_{111}, \lambda_{121}, \lambda_{211}, \lambda_{221}\} = \{Q\}(\lambda_{111} + \lambda_{221}) = (\lambda_{111} + \lambda_{221})\{Q\},$$

and we may have either or both of the sets

$$\{\lambda_{310}, \lambda_{320}, \lambda_{410}, \lambda_{420}\} = \{Q\}(\lambda_{310} + \lambda_{420}) = (\lambda_{310} + \lambda_{420})\{Q\},$$

$$\{\lambda_{131}, \lambda_{141}, \lambda_{231}, \lambda_{241}\} = \{Q\}(\lambda_{131} + \lambda_{241}) = (\lambda_{131} + \lambda_{241})\{Q\}.$$

These examples show that the algebras are equivalent to the "product" of $\{Q\}$, the quaternion system, into the systems

110		110		
111	or	111	110	121
112		210		220
...				

respectively.

A theorem given by SCHEFFERS* follows easily and as a matter of course. As may be seen at once it is easily extended to the classes of algebras that are higher in character, viz. *nonionic*, and *matrical*, in general. This is a very

* Loc. cit., S. 374.

remarkable generalization, stated, it is believed, for the first time. It may be stated thus:

Let there be given a certain frame corresponding to a set of numbers Q; let there be also given a frame corresponding to a set of numbers P; let there be constructed a frame S by substituting for each λ_{ii0} in P, the frame (square matrix) for Q, writing this matrix with different subscripts for λ_{110}, λ_{220}, $\lambda_{330}, \cdots, \lambda_{pp0}$; also let square matrices be substituted for every λ_{ijk}, each similar in structure to Q, and each deriving its first subscripts from those in λ_{ii0}, its second subscripts from those in λ_{jj0}, its third subscripts by the addition of k to corresponding third subscripts in λ_{ii0} or λ_{jj0}; finally, let the shear factors of the diagonal of S coalesce, then is

$$\{S\} = \{P\} \cdot \{Q\},$$

and conversely.

Thus, the first case above is derived from

$$Q = \left\{ \begin{matrix} 110 & 120 \\ 210 & 220 \end{matrix} \right\}, \qquad P = \left\{ \begin{matrix} 110 & & \\ 111 & 110 & \\ 112 & 111 & 110 \end{matrix} \right\};$$

from the second case

$$Q = \left\{ \begin{matrix} 110 & 120 \\ 210 & 220 \end{matrix} \right\}, \qquad P = \left\{ \begin{matrix} 110 & & \\ 111 & 110 & 121 \\ 211 & & 220 \end{matrix} \right\},$$

This theorem, however, applies to frames P and Q of *any* character.

§ 4. Algebras whose frames have factors of width three, might be called *nonionic*, and if the factors are *any form of matrix, matrical*; thus we may have algebras with the equation

$$\begin{vmatrix} a_{110} - \phi & a_{120} & a_{130} \\ a_{210} & a_{220} - \phi & a_{230} \\ a_{310} & a_{320} & a_{330} - \phi \end{vmatrix}^2 (a_{440} - \phi)^3 = 0,$$

or with

$$\begin{vmatrix} a_{110} - \phi & a_{120} & 0 & 0 \\ a_{210} & a_{220} - \phi & a_{230} & 0 \\ a_{310} & a_{320} & -\phi & a_{340} \\ 0 & 0 & a_{430} & a_{440} - \phi \end{vmatrix}^2 = 0.$$

§ 5. The associative units arising from the main diagonal squares, or from the shear factors, in other words, form sub-algebras. These are MOLIEN's "begleitende" and "ursprüngliche" * number systems. MOLIEN's theorems do not touch the other units corresponding to factors of multiplicity more than unity.

It was C. S. PEIRCE who first demonstrated the matrix character of linear associative algebra.† The work of BENJAMIN PEIRCE was first put forth in 1870. It was the first to prove the existence of the different idempotent units, and the consequent separation of an algebra into sets of units on the basis of these idempotent units. The idempotents are the η's of SCHEFFERS, and the "gerade" units are the units of PEIRCE's first or fourth, the "schiefe" of the second or third groups, respectively. C. S. PEIRCE seems to have made little mathematical use of his theorem that all algebras are modified forms of "relative" algebras.

§ 6. We close with a general theorem in extension of one given by PEIRCE.

By differentiating the equation of an algebra and remembering that $d\phi$ is any number, we arrive at equations which the numbers of the algebra must satisfy.

For instance, let the equation be

$$(a_{110}-\phi)^2=0,$$

then

$$(a_{110}-\phi)(a'_{110}-\phi')+(a'_{110}-\phi')(a_{110}-\phi)=0$$

is true of any two numbers ϕ, ϕ' of the algebra.

Let the equation be

$$(a_{110}-\phi)^2(a_{220}-\phi)=0,$$

then

$$(a_{110}-\phi)(a'_{110}-\phi')(a_{220}-\phi)+(a'_{110}-\phi')(a_{110}-\phi)(a_{220}-\phi)$$
$$+(a_{110}-\phi)^2(a'_{220}-\phi')=0,$$

and

$$(a''_{110}-\phi'')(a'_{110}-\phi')(a_{220}-\phi)+(a_{110}-\phi)(a'_{110}-\phi')(a''_{220}-\phi'')$$
$$+(a'_{110}-\phi')(a''_{110}-\phi'')(a_{220}-\phi)+(a'_{110}-\phi')(a_{110}-\phi)(a''_{220}-\phi'')$$
$$+(a''_{110}-\phi'')(a_{110}-\phi)(a'_{220}-\phi')+(a_{110}-\phi)(a''_{110}-\phi'')(a'_{220}-\phi')=0.$$

Corollary. In the case

$$\phi^n=0,$$

$$\phi_1\phi_2\cdots\phi_n+\phi_2\phi_1\cdots\phi_n+\cdots+\phi_1\phi_2\cdots\phi_n=0,$$

all permutations of the numbers appearing.

This corollary was first stated by BENJAMIN PEIRCE.

---

* *Ueber Systeme höherer complexer Zahlen*, Mathematische Annalen, vol. 41 (1892), p. 83–156.

† Proceedings American Academy of Arts and Sciences, May 11, 1875. See also *Linear Associative Algebra*, American Journal of Mathematics, vol. 4, p. 97.

In a manner quite similar to the development of quaternions, combinations of the numbers may be named and studied, further developing the calculus side of the theory, as

$$\phi_2\phi_1 - \phi_1\phi_2, \text{ etc.}$$

§ 7. As examples, the following cases of SCHEFFERS are expressed in the forms given here.

II_1. $\lambda_{110}, \lambda_{111},$ $[(a_{110}-\varphi)^2=0]$.

III_1. $\lambda_{110}, \lambda_{111}, \lambda_{112},$ $[(a_{110}-\varphi)^3=0]$.

III_2. $\lambda_{110}, \lambda_{220}, \lambda_{210},$ $[(a_{110}-\varphi)(a_{220}-\varphi)=0]$.

III_3. $\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{210},$ $[(a_{110}-\varphi)^2=0]$.

IV_1. $\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113},$ $[(a_{110}-\varphi)^2=0]$.

IV_2. $\lambda_{110}, \lambda_{220}, \lambda_{111}, \lambda_{121},$ $[(a_{110}-\varphi)^2(a_{220}-\varphi)=0]$.

IV_3. $\lambda_{110}+\lambda_{220}, \frac{1}{2}(\lambda+1)\lambda_{210}+2\cdot\lambda_{122}, \lambda_{111}+\lambda_{210}, \lambda_{112},$ $[(a_{110}-\varphi)^3=0]$.

IV_4. $\lambda_{110}+\lambda_{220}, \lambda_{111}, b\lambda_{210}+\frac{1}{b}\lambda_{122}, \lambda_{112},$ $[(a_{110}-\varphi)^3=0]$.

IV_5. $\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{210}, \lambda_{112},$ $[(a_{110}-\varphi)^3=0]$.

IV_6. $\lambda_{110}, \lambda_{220}, \lambda_{210}, \lambda_{211},$ $[(a_{110}-\varphi)(a_{220}-\varphi)=0]$.

IV_7. $\lambda_{110}, \lambda_{220}, \lambda_{211}, \lambda_{121},$ $[(a_{110}-\varphi)(a_{220}-\varphi)=0]$.

IV_8. $\lambda_{110}+\lambda_{220}, \lambda_{111}-\lambda_{221}, -\lambda_{210}, \lambda_{211},$ $[(a_{110}-\varphi)^2=0]$.

IV_9. $\lambda_{110}+\lambda_{220}, \lambda_{111}+\lambda_{221}, \lambda_{211}, \lambda_{121},$ $[(a_{110}-\varphi)^2=0]$.

IV_{10}. $\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220},$ $\left[\begin{vmatrix} a_{110}-\varphi & a_{120} \\ a_{210} & a_{220}-\varphi \end{vmatrix}=0\right]$.

V_1. $\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{114},$ $[(a_{110}-\varphi)^5=0]$.

V_2. $\lambda_{110}, \lambda_{220}, \lambda_{111}, \lambda_{112}, \lambda_{122},$ $[(a_{110}-\varphi)^3(a_{220}-\varphi)=0]$.

V_3. $\lambda_{110}, \lambda_{220}, \lambda_{111}, \lambda_{221}, \lambda_{210},$ $[(a_{110}-\varphi)^2(a_{220}-\varphi)^2=0]$.

V_4. $\lambda_{110}+\lambda_{220}, \lambda_{210}+\lambda_{123}-\lambda_{112}, \lambda_{111}+2\cdot\lambda_{123}, \lambda_{112}, \lambda_{113},$ $[(a_{110}-\varphi)^4=0]$.

V_5. $\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{210}+\lambda_{123},$ $[(a_{110}-\varphi)^4=0]$.

V_6. $\lambda_{110}+\lambda_{220}, \lambda_{111}+2\lambda_{123}, \lambda_{112}, \lambda_{113}, \lambda_{211},$ $[(a_{110}-\varphi)^4=0]$.

V_7. $\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{210},$ $[(a_{110}-\varphi)^4=0]$.

V_8. $\lambda_{110}, \lambda_{220}, \lambda_{330}, \lambda_{310}, \lambda_{320},$ $[(a_{110}-\varphi)(a_{220}-\varphi)(a_{330}-\varphi)=0]$.

V_9. $\lambda_{110}, \lambda_{220}, \lambda_{330}, \lambda_{210}, \lambda_{320},$ $[(a_{110}-\varphi)(a_{220}-\varphi)(a_{330}-\varphi)=0]$.

V_{10}. $\lambda_{110}, \lambda_{220}, \lambda_{221}, \lambda_{211}, \lambda_{210},$ $[(a_{110}-\varphi)(a_{220}-\varphi)^2=0]$.

V_{11}. $\lambda_{110}, \lambda_{112}, \lambda_{220}, \lambda_{121}, \lambda_{122},$ $[(a_{110}-\varphi)^2(a_{220}-\varphi)=0]$.

V_{12}. $\lambda_{110}, \lambda_{220}, \lambda_{210}, \lambda_{122}, \lambda_{112},$ $[(a_{110}-\varphi)^2(a_{220}-\varphi)=0]$.

V_{13}. $\lambda_{110}, \lambda_{220}, \lambda_{211}, \lambda_{122}, \lambda_{112},$ $[(a_{110}-\varphi)^2(a_{220}-\varphi)=0]$.

§ 8. As further examples, the following of PEIRCE's cases are expressed in the present form:

(a_1) $\lambda_{110},$ $[\varphi - xi = 0]$.

(b_1) $\lambda_{111},$ $[\varphi = 0]$.

(a_2) $\lambda_{110}, \lambda_{111},$ $[(\varphi - xi)^2 = 0]$.

(b_2) $\lambda_{110}, \lambda_{120},$ $[(\varphi - xi)\varphi = 0]$.

(c_2) $\lambda_{111}, \lambda_{112},$ $[\varphi^2 = 0]$.

(a_3) $\lambda_{110}, \lambda_{111}, \lambda_{112},$ $[(\varphi - xi)^3 = 0]$.

(b_3) $\lambda_{111}, \lambda_{112}, \lambda_{113},$ $[\varphi^3 = 0]$.

(c_3) $a\lambda_{210} + \lambda_{111}, \lambda_{112}, \lambda_{210} + \lambda_{122},$ $[\varphi^3 = 0]$.

(d_3) $\lambda_{210} + \lambda_{111}, \lambda_{112}, \lambda_{122},$ $[\varphi^3 = 0]$.

(e_3) $\lambda_{211}, \lambda_{210}, \lambda_{111} - \lambda_{121},$ $[\varphi^2 = 0]$.

(a_4) $\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113},$ $[(\varphi - xi)^4 = 0]$.

(b_4) $\lambda_{110} + \lambda_{220}, a\lambda_{210} + \lambda_{111}, \lambda_{112}, \lambda_{210} + \lambda_{122},$ $[(\varphi - xi^3) = 0]$.

(c_4) $\lambda_{110} + \lambda_{220}, \lambda_{210} + \lambda_{111}, \lambda_{112}, \lambda_{122},$ $[(\varphi - xi)^3 = 0]$.

(d_4) $\lambda_{110} + \lambda_{220}, \lambda_{211}, \lambda_{210}, \lambda_{111} - \lambda_{121},$ $[(\varphi - xi)^2 = 0]$.

(e_4) $\lambda_{220}, \lambda_{221}, \lambda_{210}, \lambda_{211},$ $[(\varphi - xi)^2\varphi = 0]$.

(f_4) $\lambda_{220}, \lambda_{222}, \lambda_{212}, \lambda_{120},$ $[(\varphi - xi)^3 = 0]$.

(g_4) $\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220},$ $\left[\begin{vmatrix} x_1 i - \varphi & x_2 \\ x_3 & x_4 l - \varphi \end{vmatrix} = 0\right]$.

(h_4) $\lambda_{110}, \lambda_{120}, \lambda_{212}, \lambda_{222},$ $[(\varphi - xi)\varphi^3 = 0]$.

(i_4) $\lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{114},$ $[\varphi^4 = 0]$.

The completion of these tables is a simple matter. It is proposed in a following paper to extend them and systematize them.

§ 9. A natural classification of algebras seems to be that according to the number and character of distinct factors. Thus we have algebras of class:

linear matrix, all factors linear;
quadratic matrix, some factors determinants of second order, quadratic-linear, quadratic-quadratic, etc.;
cubic matrix, etc.;
of degree:
one, *two*, *three*, according to the number of factors:
of multiplicity:
one, *two*, *three*, according to the powers of factors;

A natural treatment seems to be first to consider those of equation

$$(a-\phi)^m = 0,$$

second, of equation

$$(a_1-\phi)^{m_1}(a_2-\phi)^{m_2}\cdots = 0,$$

third, of equation

$$\begin{vmatrix} a_{11}-\phi & a_{12} \\ a_{21} & a_{22}-\phi \end{vmatrix}^m = 0,$$

etc.

This method we might call the *statistical classification*.

A second classification is according to the order of the framework necessary. Thus for a framework of order 2, we may have

$$\begin{array}{|cc|}\hline \cdot & \cdot \\ \cdot & \cdot \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \cdot \\ \cdot \quad \cdot \\ \hline \end{array}$$

(1) (2)

(1) $\lambda_{110},\ \lambda_{120},\ \lambda_{210},\ \lambda_{220},$

with sub-cases

(11) $\lambda_{110},\ \lambda_{220},\ \lambda_{120},$

(12) $\lambda_{110},\ \lambda_{220},\ \lambda_{210},$

(13) $\lambda_{110},\ \lambda_{220},$

(14) $\lambda_{110},\ \lambda_{120},$

(15) $\lambda_{110},$

(16) $\lambda_{120},$

(2) $\lambda_{110}+\lambda_{220},\ \lambda_{210},$

(3) $\lambda_{110},\ \lambda_{111}.$

In such cases we should frequently find algebras of units whose substitution effect on the *framework* would be different, yet whose combination tables would be the same.

September 1, 1902.

PROJECTIVE COÖRDINATES*

BY

F. MORLEY

The definition of coördinates in projective geometry should naturally be projectively stated. There is, I think, a very distinct advantage in taking them as double ratios, a step which I do not find in the literature. †

The matter, restricted to two dimensions for simplicity of statement only, might be presented as follows.

I assume that the theory in one dimension has been explained, and that the notion of the double ratio of two points xy and two lines $\xi\eta$ of a plane is known; on varying the order there are *two* reciprocal double ratios

$$(xy\,|\,\xi\eta) \qquad \text{and} \qquad (yx\,|\,\xi\eta).$$

§ 1. *Point coördinates.*

Let now four lines α_0, α_i $(i = 1, 2, 3)$ of a plane be given; call α_0 the *auxiliary line* and the triangle formed by the other 3 lines the triangle of reference. Let its points be a_1, a_2, a_3.

With any point x of the plane, not on α_0, are associated 3 double ratios

$$r_i = (xa_i\,|\,\alpha_i\alpha_0). \tag{1}$$

These three double ratios are the projective coördinates of the point.

When the auxiliary line is the line at infinity, we have the barycentric case; it appears at once from this metrically canonic case that always

$$r_1 + r_2 + r_3 = 1. \tag{2}$$

That is,

The sum of the projective coördinates is 1.

When one coördinate of x is given, x lies on a line; and any two coördinates determine the point. But for points on the auxiliary line α_0 the coördinates are infinite. See the last paragraph of § 4.

Ordinary Cartesian coördinates require no peculiar treatment; the triangle of reference is formed by the axes and the line joining the unit points on the

* To be presented to the Society at the Boston summer meeting, August 31–September 1, 1903. Received for publication April 27, 1903.

† The nearest approach that I know of is made by KOHN, Wiener Sitzungsberichte, vol. 104 (1895), p. 1167.

axes, the auxiliary line being at infinity. The only difference between barycentric and cartesian coördinates lies in the fact that we usually assume in the latter case an isosceles triangle of reference.

§ 2. *Line coördinates.*

With the same reference triangle take an auxiliary point a_0, and with a line ξ associate the double ratios

$$\rho_i = (\xi\alpha_i \mid a_i a_0). \tag{3}$$

For a metrically canonic case we take a_0 to be the centroid of the reference triangle. Then if δ_j be the distance from a_j to ξ, $(j = 0, \cdots, 3)$,

$$\rho_i = \tfrac{1}{3}\frac{\delta_i}{\delta_0}.$$

We have then *always*,

$$\rho_1 + \rho_2 + \rho_3 = 1. \tag{4}$$

except for lines through a_0, for which ρ_i is infinite.

§ 3. *Point and line.*

In the barycentric case we placed the auxiliary line α_0 at infinity, the auxiliary point a_0 at the centroid. So always we take a_0 as the polar point of α_0 as to the reference triangle. The well-known properties of polar point and line are read off from the barycentric case. Thus:

The polars of a line α_0 as to 2 of 3 points a_i form a triangle perspective with the triangle a, and the perspective point is the polar of α_0. Or:

If a 3-point is incident with a 3-line (i. e., each point on a line) and the two triangles are perspective, then the perspective point and line are polar point and line as to either triangle.

Now in the barycentric case the distance from the point x to the line ξ is

$$\delta = r_1\delta_1 + r_2\delta_2 + r_3\delta_3 = (r\delta).$$

Hence

$$\tfrac{1}{3}\frac{\delta}{\delta_0} = \left(r, \tfrac{1}{3}\frac{\delta}{\delta_0}\right).$$

Hence in general (§ 2) the double ratio $(xa_0 \mid \xi\alpha_0)$ is given by

$$\tfrac{1}{3}(xa_0 \mid \xi\alpha_0) = (r\rho) \tag{5}$$

and in particular *when a point and line are incident*

$$r_1\rho_1 + r_2\rho_2 + r_3\rho_3 = 0. \tag{6}$$

§ 4. *The symmetrical system of point coördinates.*

The double ratio $(xx' \mid \xi\xi')$ of two points xx' and two lines $\xi\xi'$ is (appealing once more to the barycentric case)

$$(xx' \mid \xi\xi') = \frac{(r\rho)(r'\rho')}{(r\rho')(r'\rho)}.$$

And in particular

$$r_1' = (xa_i \mid \alpha_1\alpha_0) = \frac{(x\alpha_1)(a_1\alpha_0)}{(x\alpha_0)(a_1\alpha_1)}.$$

It is no longer convenient to think of the triangle a_i or α_i as the triangle of reference—rather it is any triangle, the reference triangle being placed in the background. The point a_1, the join of the lines α_2 and α_3, is given by

$$(a_1\alpha_2) = 0, \qquad (a_1\alpha_3) = 0.$$

Hence

$$r_1 = \frac{(x\alpha_1)\,|\alpha_2\alpha_3\alpha_0|}{(x\alpha_0)\,|\alpha_2\alpha_3\alpha_1|},$$

and the relation

$$r_1 + r_2 + r_3 = 1$$

becomes

$$(x\alpha_1)\,|\alpha_2\alpha_3\alpha_0| + (x\alpha_2)\,|\alpha_3\alpha_1\alpha_0| + (x\alpha_3)\,|\alpha_1\alpha_2\alpha_0| = (x\alpha_0)\,|\alpha_2\alpha_3\alpha_1|,$$

or say

$$x_1 + x_2 + x_3 + x_0 = 0. \tag{7}$$

These "supernumerary homogeneous coördinates" x_j, or as I would call them *symmetrical coördinates*, are used only in connection with homogeneous equations; whereas with the double ratios it is mainly as a matter of convenience in polarizing that we prefer homogeneous equations.

It will be noticed that the symmetrical system is not intuitively a projective system at all; that is, we do not have (at all events at present) a direct projective significance for x_j expressed in terms of the 4 lines and the point alone, but only for a ratio of two x_j's.

In passing from 3 dimensions where $r_i = (xa_i \mid \alpha_i\alpha_0)$ $(i = 1, 2, 3, 4)$, to a plane of reference α_i we pass at once to the projective system in that plane, but in passing to the auxiliary plane α_0 we pass at once to the symmetrical system in that plane. And so in general.

§ 5. *The symmetrical system of line coördinates.*

In the same way, given 4 points $a_j (j = 0, 1, 2, 3)$ and a line ξ, we have

$$\rho_1 = \frac{(\xi a_1)\,|a_2a_3a_0|}{(\xi a_0)\,|a_2a_3a_1|},$$

and we take the 4 numbers

$$(8)\qquad \begin{aligned} \xi_1 &= (\xi a_1)\,|\,a_2 a_3 a_0\,|,\\ \xi_2 &= (\xi a_2)\,|\,a_3 a_1 a_0\,|,\\ \xi_3 &= (\xi a_3)\,|\,a_1 a_2 a_0\,|,\\ \xi_0 &= -(\xi a_0)\,|\,a_1 a_2 a_3\,|, \end{aligned}$$

whose sum is zero, as symmetrical coördinates of the line with reference to the four points.

But the 4 points have to be connected with the 4 lines of § 4, as otherwise the geometrical use of symmetrical systems would be extremely restricted.

Any rational equation in r_i gives a curve which is of course a covariant of the 4 lines α_j; the curve simply carries its equation with it when projected. And so any homogeneous equation in x_j defines a covariant curve of the 4 lines; but of special interest are the symmetrical equations, defining curves not altered by interchange of lines.* And of these the simplest is

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

The polars of the 4 lines as to this nullipartite conic are the 4 points to be associated with the 4 lines. Let us call the two tetrads *counter-tetrads*; their combination a *frame of reference*.

§ 6. *The metrically canonic frame.*

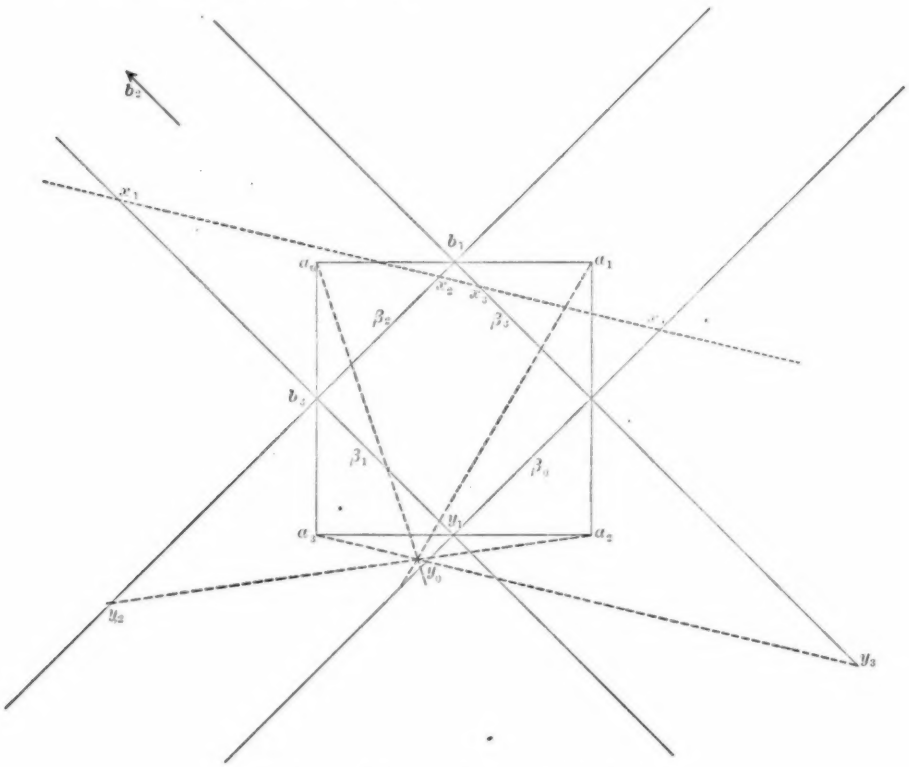
Denote by S_n a flat space of n dimensions. Let the plane be the plane at infinity in S_3; and replace the points and lines by lines and planes through an origin O. Let the nullipartite absolute of this (elliptic) geometry be given by $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$.

In the plane $x_j = 0$ we have a geometry with the intersection of that plane and the absolute cone as absolute; hence from the theory in S_1 the planes $x_1 x_2 x_3$ cut x_0 in equispaced lines; whence the four planes are parallel to the faces of a regular tetrahedron, or, say, are *equispaced*. With these planes associate their normals through O; these are the diagonals of a cube. The combination is the metrically canonic frame, and the mutuality of the tetrads is evident.

To obtain a canonic frame in a plane we may cut the frame just mentioned across a diagonal of the cube, thus getting for the four points the vertices and center of an equilateral triangle, and for one of the lines the line at infinity;

* That is, invariant under KLEIN's group of 24 collineations which leave the 4-line unaltered. KLEIN, Mathematische Annalen, vol. 4 (1871), p. 346, or *Icosaeder*, p. 166; MOORE, American Journal of Mathematics, vol. 22 (1900).

whence *each line is the polar line of one of the points as to the 3-line formed by the other points*, and reciprocally *each point is the polar point of a line as to the 3-point formed by the other lines.* Or, again, we may cut the frame by a face of the cube; we thus get the most convenient form (see figure), but it is not one that can be generalized for S_n, whereas the regular $(n+1)$-point and its center are always available in S_n.



§ 7. *Point and line in the symmetrical system.*

It is now to be shown that when a point x is referred to 4 lines and a line ξ to the counter-tetrad of 4 points, the condition of incidence is

$$(x\xi) \equiv x_0\xi_0 + x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0.$$

Let the lines be β_j, the points a_j. When referred to the triangle $\beta_1\beta_2\beta_3$ or $b_1b_2b_3$, for which the auxiliary point and line are a_0 and β_0, the projective coördinates of x are simply

$$r_i = -x_i/x_0.$$

Those of ξ are $\rho_i = (\xi\beta_i | b_i a_0)$, while $-\xi_i/\xi_0 = (\xi\alpha_i | a_2 a_0)$: Estimating along the diagonal $\overline{a_0 a_2}$ of the figure we have

$$\rho_2 = \frac{\beta_2 - a_0}{\xi - a_0},$$

$$-\xi_2/\xi_0 = \frac{x - a_2}{x - a_0} \cdot \frac{\alpha_2 - a_0}{\alpha_2 - a_2}$$

$$= -\frac{x - a_2}{x - a_0}$$

$$= -1 + 4\frac{\beta_2 - a_0}{x - a_0}$$

$$= -1 + 4\rho_2.$$

That is,

$$1 - \xi_i/\xi_0 = 4\rho_i.$$

Hence

$$4(r\rho) = 1 + \sum^3 \frac{x_i \xi_i}{x_0 \xi_0} = \frac{(x\xi)}{x_0 \xi_0}.$$

If point and line unite, then

$$(r\rho) = 0,$$

and therefore

(10) $$(x\xi) = 0.$$

If not, then

(11) $$\tfrac{1}{3}(xa_0 | \xi\beta_0) = \tfrac{1}{4}\frac{(x\xi)}{x_0 \xi_0},$$

or say

$$\tfrac{1}{3}\lambda_j = \tfrac{1}{4}\frac{(x\xi)}{x_j \xi_j},$$

whence the relation between the double ratios $\lambda_j \equiv (xa_j | \xi\beta_j)$ is

(12) $$\sum \frac{1}{\lambda_j} = \tfrac{4}{3}.$$

[And so for the double ratios formed by a point and an S_{n-1} in S_n, when referred to a frame, defined by $n+2$ points a_j and the counter-scheme of S_{n-1}'s β_j, if

$$\lambda_j = (xa_j | \xi\beta_j),$$

$$\sum \frac{1}{\lambda_j} = \frac{n+2}{n+1}.$$

For $n = 1$ we may regard a_j and β_j as complex numbers and the canonic frame is then the vertices of a regular hexagon.]

§ 8. *Like-named point and line.*

It is worth while to show how to construct the polar system

$$\xi_0/x_0 = \xi_1/x_1 = \xi_2/x_2 = \xi_3/x_3$$

which is that of the covariant conic

$$\sum x_j^2 = 0.$$

Referring to the cube-diagonals this is merely to construct the plane normal to a given line; but we wish a ruler-construction in the plane.

We suppose the problem solved for S_1; that is that the partner of a point, or its polar as to the Hessian pair of three points, has been constructed. The quickest way (though of course not a projective way) is, when the three points a_j on a line are given to draw arcs containing the angle $\pi/3$ through a_0a_1 and a_1a_2 on the same side of the line: a T-square whose vertex is placed when the arcs meet marks off on the line the required involution.

Construct then the partner of the point where a line ξ meets β_0, as to the triad of points where $\beta_1\beta_2\beta_3$ meet β_0. Join this partner y_0 to a_0. We get thus 4 lines y_ja_j which meet in the point required.

The polar system being now constructed for S_2 can be extended in the same way for S_3 and so on. That is, *the polar system of the invariant quadric is constructed.*

The figure (drawn by Mr. Carver) shows the present case $n = 2$.

§ 9. *Like-named points for two frames.*

A word on the beginning of the treatment of collineations may be added. The distinction between two views must be made—call them *alias* and *alibi*. In the former we consider the relations of the coördinates of a point x referred to different frames F, F'; in the second F' is projected into F and thereby x into x'. The like-named points of *alias* are the fixed points of *alibi*.

We ask now for the points which are like-named as to two frames F, F'. That is, the points for which

$$r_i = r_i',$$

or

$$(xa_i \mid \alpha_i\alpha_0) = (xa_i' \mid \alpha_i'\alpha_0'),$$

or, in the symmetrical system,

$$x_i = \kappa x_i'.$$

They are the points common to all the conics

$$\frac{x_0}{x_0'} = \frac{x_1}{x_1'} = \frac{x_2}{x_2'} = \frac{x_3}{x_3'}. \tag{13}$$

The general case is that of three distinct intersections. If these like-named points are taken as points of both frames, then the "change of coördinates" is merely a change of auxiliary point and therefore of auxiliary line. Particularly important is the case when the new auxiliary point lies on the old auxiliary line; for then if

$$r_i = \kappa_i r_i'$$

be the transformation we have $\kappa_i / \sum \kappa_i$ on the line for which r_i is ∞, so that

$$\sum \kappa_i = 0,$$

and the collineation is *normal*.

§ 10. *Hun's theorem.*

It was shown by PASCH* that, given a normal collineation N of a plane which sends any figure F into NF, there are triangles T inscribed in NT. In fact if two points ab become Na, Nb and the joins of a and Nb, b and Na determine c, then Nc is on ab.

There are, for given fixed points of N, ∞^5 such PASCH triangles. Mr. HUN has proved that *the relation of a Pasch triangle and the triangle of the fixed points is mutual.* I wish to prove this fact in such form that the restriction to two dimensions falls away. The fixed points being reference points, N is of the form

$$\kappa_1 x_1 \xi_1 + \kappa_2 x_2 \xi_2 + \kappa_3 x_3 \xi_3 = 0, \tag{14}$$

where

$$\kappa_1 + \kappa_2 + \kappa_3 = 0.$$

And

$$x_1 \xi_1 = (x a_1 | \alpha_1 \alpha_0)(\xi \alpha_1 | a_1 a_0)$$

$$= \frac{(x a_1)(a_1 \alpha_0)\ (\xi a_1)(\alpha_1 a_0)}{(x \alpha_0)(a_1 \alpha_1)\ (\xi a_0)(\alpha_1 a_1)}.$$

Hence (14) takes the form

$$\Sigma \lambda_i (x \alpha_1)(\xi a_i) = 0,$$

so that if there be a triangle x_i, ξ_i of the PASCH kind (in which case the collineation is normal) then

---

*Mathematische Annalen, vol 23 (1884).

$$\begin{vmatrix} (x_1\alpha_1)(\xi_1 a_1) & (x_1\alpha_2)(\xi_1 a_2) & (x_1\alpha_3)(\xi_1 a_3) \\ (x_2\alpha_1)(\xi_2 a_1) & (x_2\alpha_2)(\xi_2 a_2) & (x_2\alpha_3)(\xi_2 a_3) \\ (x_3\alpha_1)(\xi_3 a_1) & (x_3\alpha_2)(\xi_3 a_2) & (x_3\alpha_3)(\xi_3 a_3) \end{vmatrix} = 0,$$

or better, dividing rows by $(x_i\xi_i)$ and columns by $(\alpha_i a_i)$,

$$|(x_i a_k|\alpha_k \xi_i)| = 0 \qquad (i, k=1, 2, 3). \tag{15}$$

The mutuality is now evident; and the extension to n dimensions is also evident, we have merely $i, k = 1, 2, \cdots, n+1$. The determinant contains an extraneous factor which vanishes if the $n+1$ points x_i or a_i lie in an S_{n-1} or the $n+1$ S_{n-1}'s lie in a point. In fact the sum of the elements of the determinants in any row or column is 1.

Thus, the determinant being written $|r_{ik}|$, the condition is, for $n = 1$,

$$\begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} = 0, \qquad \text{or} \qquad \begin{vmatrix} r_{11} & 1 \\ 1 & 2 \end{vmatrix} = 0;$$

and, for $n = 2$,

$$\begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} = 0, \qquad \text{or} \qquad \begin{vmatrix} r_{11} & r_{12} & 1 \\ r_{21} & r_{22} & 1 \\ 1 & 1 & 3 \end{vmatrix} = 0.$$

JOHNS HOPKINS UNIVERSITY,
February, 1903.

ON AN EXTENSION OF THE 1894 MEMOIR OF STIELTJES*

BY

EDWARD BURR VAN VLECK

INTRODUCTION

In the closing memoir† of his life STIELTJES gave an elegant and exhaustive discussion of the continued fraction

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \cdots, \tag{1}$$

in which the elements a_n are all positive. To this continued fraction he showed that there corresponds, on the one hand, a single series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_3}{z^3} - \frac{c_4}{z^4} + \cdots \tag{2}$$

in which the coefficients are conditioned by the relations

$$A_n \equiv \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} > 0, \qquad B_n \equiv \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix} > 0,$$

and, on the other hand, one or more functional integrals

$$\int_0^\infty \frac{d\Phi(u)}{z+u} \tag{3}$$

in which $\Phi(u)$ is a real and increasing function of u, though not necessarily analytic or continuous. Conversely, to every such series or integral (provided the integral permits of formal expansion into a series of the form (2)) there corre-

* Presented to the Society, February 28, 1903. Received for publication, May 25, 1903.

† Annales de la Faculté des Sciences de Toulouse, vol. 8 (1894), with continuation in vol. 9. The continued fraction of STIELTJES is also treated in BLUMENTHAL'S Dissertation: *Ueber die Entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruches für*

$$\int_{-\infty}^{0} \frac{\Phi(\xi)d\xi}{z-\xi}.$$

Göttingen, 1898.

sponds a single continued fraction of the form (1), and the elements of this continued fraction are positive.

In the extension of the work of STIELTJES which I shall give here the amount of restriction imposed upon the continued fraction and series will be diminished by half. In the former it will be required either that $a_{2n+1} > 0$ or that $a_{2n} > 0$. The corresponding conditions for the series are that $A_n > 0$, $B_n \neq 0$ or $B_n > 0$, $A_n \neq 0$. If, furthermore, the even convergents in the one case and the odd convergents in the other are considered in separate continued fractions instead of being united by (1) into a single continued fraction, the subsidiary conditions $A_n \neq 0$ and $B_n \neq 0$ may be dispensed with entirely.

The integrals which correspond to the series and continued fractions, in case they exist, are of the form

$$\int_b^a \frac{d\Phi(u)}{z+u} \qquad (-\infty \leqq b < a \leqq +\infty),$$

$\Phi(u)$ being of the same character as in (3). The change thus made in the limits of integration brings the work of STIELTJES into close connection with integrals considered by HEINE, TSCHEBYSCHEFF, and others.

If the continued fraction (1) is thrown with the aid of the substitution $z = 1/x$ into the more familiar form

$$\cfrac{b_1 x}{1} + \cfrac{b_2 x}{1} + \cfrac{b_3 x}{1} + \cdots \qquad \left(b_n = \frac{1}{a_n a_{n-1}}\right),$$

the restrictions upon the coefficients are that b_{2n} and b_{2n+1} (or b_{2n+1} and b_{2n+2}) shall always have a common sign, the sign, however, being arbitrary for each value of n.

The condition imposed by STIELTJES upon the coefficients of (1) seems to be the simplest one possible. Not only does it suffice to ensure the reality of the roots of the numerators and denominators of the convergents, but certain theorems* also follow concerning the alternation of the roots when two convergents are compared. On these theorems STIELTJES rears his structure. In Section I of the present paper I have shown that these theorems will also hold for the more general class of continued fractions here treated, if the positive and negative halves of the real axes are considered apart.

In most particulars the theory of these continued fractions is parallel to that of STIELTJES, and his line of thought can be modified so as to apply. The greatest differences between the two theories appear in matters of convergence. STIELTJES distinguishes two cases of his continued fraction. When $\sum_{n=1}^{\infty} a_n$ is divergent, the continued fraction converges over the entire plane of z with the exception of a part or the whole of the negative half of the real axis, and its

* Loc. cit., §§ 3–5.

limit is an analytic function which can be expressed by means of an integral of the form (3). The corresponding series does not, however, necessarily converge. If, in short, c_n/c_{n-1}, which increases with n, increases beyond all limit, the series is divergent. A special interest attaches to this case, because the divergent series still determines uniquely a convergent continued fraction and therefore an analytic function.

When $\sum a_n$ is convergent, the continued fraction itself does not converge, but the even and odd convergents separately do, thereby defining two functions. STIELTJES shows that the numerators and denominators of the two sets of convergents also have limits, and these limits are holomorphic functions of *genre* 0. The corresponding series (2) is, of course, divergent.

For the generalized continued fractions before us the matter is far more complicated. No necessary and sufficient test for convergence has been found, and it seems quite probable that no such test is possible. Criteria are, however, given which will cover most of the cases likely to arise. Whenever the continued fraction converges uniformly in any region, the region of convergence covers the entire plane of z with the exception of the whole or a part of the real axis, and the limit is holomorphic within this region. If the whole axis is to be excluded, it does not follow of necessity that the analytic functions determined in the two half planes are distinct, but it is interesting to note that such a case does actually occur. The infinite series which corresponds to the continued fraction is divergent, and we have therefore the remarkable phenomenon, so far as I know never previously pointed out, that a divergent power series may define two analytic functions separated by a natural boundary.

In addition to the two special cases of the continued fraction distinguished by STIELTJES one other may be mentioned, in which the alternate even convergents form two separate sequences. In either sequence the limits of the numerator and denominator are holomorphic functions of *genre* 1 or 0. The odd convergents, on the other hand, constitute a single sequence of the same character.

The last section of the paper is devoted to an examination of the table of rational approximants for the series (2), constructed after the manner of PADÉ. The table is "normal" for the continued fraction of STIELTJES and contains only irregularities of the first order when $A_n > 0$ or $B_n > 0$. The paper closes with a brief consideration of the zeros of the numerators and denominators of the approximants. The most interesting result here reached is published separately* and relates to the approximant obtained by taking the first $2n + 2$ or $2n + 1$ terms of (2). This polynomial has the maximum number of imaginary roots, $2n$, thus differing essentially from the approximants used as the convergents of (1), the zeros and infinities of which are all real.

*Annals of Mathematics, July, 1903.

No attempt is made to give a complete discussion of all the questions which arise in the extension of the memoir of STIELTJES, but much is left for subsequent investigation.

I. ON THE ZEROS AND INFINITIES OF THE CONVERGENTS OF THE CONTINUED FRACTION.

1. *The formation of a Sturm series.*—Let N_n and D_n denote the numerator and denominator of the nth convergent of a continued fraction

$$\frac{1}{1+}\ \frac{b_2 x}{1}\ +\ \frac{b_3 x}{1}\ +\ \cdots \tag{4}$$

in which the elements b_n are real. From the fundamental relations

$$\begin{aligned} D_n &= D_{n-1} + b_n x D_{n-2}, \\ N_n &= N_{n-1} + b_n x N_{n-2}, \end{aligned} \tag{5}$$

it can be shown at once by mathematical induction that D_n and N_n have for an even subscript the form

$$\begin{aligned} D_{2n} &= 1 + (b_2 + b_3 + b_4 + \cdots + b_{2n})x + \cdots + b_2 b_4 \cdots b_{2n} x^n, \\ N_{2n} &= 1 + (b_3 + b_4 + \cdots + b_{2n})x + \cdots + (\quad) x^{n-1}, \end{aligned} \tag{6}$$

and for an odd subscript

$$\begin{aligned} D_{2n+1} &= 1 + (b_2 + b_3 + \cdots + b_{2n+1})x + \cdots + (\quad) x^n, \\ N_{2n+1} &= 1 + (b_3 + b_4 + \cdots + b_{2n+1})x + \cdots + b_3 b_5 \cdots b_{2n+1} x^n. \end{aligned} \tag{7}$$

From (5) we obtain further

$$D_n = (1 + b_{n-1}x + b_n x)\, D_{n-2} - b_{n-1} b_{n-2} x^2 D_{n-4} \tag{8}$$

and a similar formula for N_n.

The last equation gives immediately a necessary condition that the alternate denominators (or numerators) of the convergents shall form a Sturm series. Since D_n and D_{n-4} must have opposite signs when D_{n-2} vanishes, either $b_{2n-1} b_{2n}$ or $b_{2n} b_{2n+1}$ must be positive for all values of n. It is really immaterial which supposition we take. For should b_{2n} and b_{2n+1} have like signs, by dropping the first partial fraction of (4) we obtain another continued fraction in which the $(2n-1)$th and $2n$th coefficients have a common sign. Now the denominators of the latter continued fraction are the numerators of the convergents of the former. The results of either supposition can therefore be derived

readily from those of the other. As also the consequences of the two suppositions are in every way parallel, it will suffice to develop here only one of the two.

We will accordingly suppose that b_{2n} and b_{2n+1} are always of the same sign, the sign being arbitrary for each value of n. Consider then the sequence

$$D_0 = 1, D_2, D_4, \cdots, D_{2n}. \tag{9}$$

When $x = 0$, all of its terms are positive, each being equal to 1. Moreover, no two consecutive terms can vanish for the same value of x, since by (8) they all would vanish simultaneously. If, then, x traverses the positive or negative half-axis, alterations in the number of changes of sign in (9) can take place only when x passes through a root of D_{2n}. As x approaches ∞, the dominant term in D_{2n} is $b_2 b_4 \cdots b_{2n} x^n$. Hence for $x = +\infty$ the number of changes of sign in (9) is equal to the number of negative signs in

$$b_2, b_4, \cdots, b_{2n}, \tag{10}$$

while for $x = -\infty$ it is equal to the number of positive signs. But the number of terms in (10) is exactly equal to the degree of D_{2n}. We have therefore

THEOREM I. — *If in the continued fraction* (4) $b_{2n} b_{2n+1}$ *is always positive, the roots of* D_{2n} *are real, and the number of positive and of negative roots is equal respectively to the number of negative and of positive terms in* (10).

2. *The even convergents.* — The considerations just adduced show also that the sequence (9) has all the properties of a Sturm series in the positive and negative half axes taken separately. As x passes out from the origin, there is a gain in the number of changes of sign in (9) every time that x passes through a root of D_{2n}. This can only occur if between two consecutive roots of D_{2n} in each half axis, there is included a root of D_{2n-2}. Hence, since all the roots of D_{2n} are real, the roots of D_{2n-2} must be likewise real and alternate with those of D_{2n} on either side of the origin. It is evident also that D_{2n} must be the first to vanish after leaving the origin.

The reality and location of the roots of N_{2n} can next be ascertained. From (8) and the analogous equation for N_n we obtain

$$D_{2n} N_{2n-2} - N_{2n} D_{2n-2} = b_{2n-1} b_{2n-2} x^2 (D_{2n-2} N_{2n-4} - N_{2n-2} D_{2n-4});$$

whence it follows that

$$D_{2n} N_{2n-2} - N_{2n} D_{2n-2} = -b_{2n-1} b_{2n-2} \cdots b_2 x^{2n-2}.$$

Let now any two consecutive roots of D_{2n} which lie on the same side of the origin be substituted in the last equation. Inasmuch as the right hand member is always negative, each root will give to $N_{2n} D_{2n-2}$ a positive sign. Since also there

is included between them one and only one root of D_{2n-2}, it follows that they must enclose one of the $n-1$ roots of N_{2n}. If the roots of D_{2n} all lie on the same side of the origin, there are $n-1$ such pairs of roots, and the location of the roots of N_{2n} will therefore have been completely determined. On the other hand, if the roots of D_{2n} lie on both sides of the origin, the number is $n-2$, and there remains accordingly one root of N_{2n} whose position is yet to be determined. Now neither of the two intervals between the origin and the adjacent roots of D_{2n} contains a root of D_{2n-2}. At the two latter points we know that $N_{2n}D_{2n-2}$ is positive, and at the origin it is also positive, being equal to 1. The remaining root of N_{2n} therefore cannot be contained in either of these two intervals, and the only place remaining for this root is between one of the two outermost roots of D_{2n} and the point of infinity.

The results thus obtained can be stated as

THEOREM II.—*On each side of the origin the roots of D_{2n} are separated by the roots of D_{2n-2} and also by those of N_{2n}, and the root nearest to the origin on either side must be a root of D_{2n}.*

3. *The odd convergents.*—We next consider the odd convergents. Between these and the even convergents there exist the familiar relations:

$$D_{2n}N_{2n-1} - N_{2n}D_{2n-1} = b_2 b_3 \cdots b_{2n} x^{2n-1}, \tag{11}$$

$$D_{2n+1}N_{2n} - N_{2n+1}D_{2n} = -b_2 b_3 \cdots b_{2n+1} x^{2n}. \tag{12}$$

If in each of these equations two consecutive roots of D_{2n} are substituted which lie upon the same side of the origin, we conclude, by reasoning in the same manner as in § 2, that they must enclose at least one root of D_{2n-1} and one of D_{2n+1} since they enclose a single root of N_{2n}. Now the degrees of D_{2n-1}, D_{2n} and D_{2n+1} are $n-1$, n, and n respectively. Hence the location of all the roots of D_{2n-1} and D_{2n+1} will be fixed relatively to those of D_{2n} with the exception of one root of the former and two of the latter, unless all the roots of D_{2n} should fall upon the same side of the origin when there would be only a root of D_{2n+1} to locate.

To fix these roots, consider again the intervals between the origin and the adjacent roots of D_{2n}. The substitution, in turn, of these two roots in (11) gives opposite signs to the right hand member and hence also to $N_{2n}D_{2n-1}$. Since neither interval contains a root of N_{2n}, it follows that one of them must contain the remaining root of D_{2n-1}. We conclude accordingly that the roots of D_{2n} and D_{2n-1} alternate along the real axis. If the same two roots are substituted in (12), $D_{2n+1}N_{2n}$ will have a negative sign in both cases since the right hand member is negative for any real value of x. But at the origin $D_{2n+1}N_{2n} = 1$. Consequently, as neither of the two intervals contains a root of N_{2n}, they must each contain a root of D_{2n+1}. A similar conclusion holds for

the single interval terminating in the origin when the roots of D_{2n} lie on only one side of the origin. This completes the proof of the following result:

THEOREM III.—*The roots of D_{2n} and D_{2n-1} alternate along the real axis, but the roots of D_{2n} and D_{2n+1} alternate only in the positive and negative half axes taken separately. In each half axis there is the same number of roots of D_{2n+1} as of D_{2n}, and the root adjacent to the origin is always a root of D_{2n+1}.*

Having ascertained now the situation of the roots D_{2n+1}, we can next employ (12) for the purpose of finding the location of the roots of N_{2n+1} with respect to those of D_{2n+1}. On the left hand side of this equation the highest power of x is obtained from the multiplication of the terms of highest degree in D_{2n} and N_{2n-1}. An inspection of (6) and (7) shows at once that the coefficients of these terms cannot vanish since $b_n \neq 0$. Hence the sign of the left hand member of (12) for $x = \pm\infty$ is identical with that of $-N_{2n+1}D_{2n}$. But the sign of the right-hand member is always negative, and consequently $N_{2n+1}D_{2n}$ is positive both for $x = +\infty$ and for $x = -\infty$. Furthermore, if x in (12) is placed equal to any root of D_{2n+1}, again $N_{2n+1}D_{2n}$ will be positive. But we know that between any two consecutive roots of D_{2n+1} on the same side of the origin and between the outermost root on either side and the point at infinity there lies one and only one root of D_{2n}. It follows that the roots of N_{2n+1} are situated in the same manner as those of D_{2n}. This result may be recapitulated for convenience of future reference in

THEOREM IV.— *The roots of N_{2n+1} and D_{2n+1} alternate on each side of the origin. On either side there is the same number of roots of one as of the other, and the roots of D_{2n+1} lie nearer to the origin than the corresponding roots of N_{2n+1}.*

To complete the discussion it remains to determine the relative positions of the roots of N_n and N_{n-1}. This also can be done with the aid of (11) and (12). It will be found that *the roots of N_{2n} and N_{2n-1} alternate along the axis while the roots of N_{2n} and N_{2n+1} alternate only on the two sides of the origin considered separately, the roots adjacent to the origin being those of N_{2n+1}.*

As it will be necessary subsequently to treat the odd convergents apart from the even, it is desirable to know also the relative positions of the roots of the denominators of two consecutive odd convergents. This can be inferred with the aid of theorem III from the equation

$$D_{2n+1} = D_{2n} + b_{2n+1}x D_{2n-1}.$$

By substituting pairs of consecutive roots of D_{2n+1} we obtain

THEOREM V.—*The roots of D_{2n+1} and D_{2n-1} alternate with each other along the axis of x.*

4. *On the relative situation of the roots of D_n and D_{n+m}.*—We proceed next to compare D_n and D_{n+m}. Let $b_{n+1}xN_m^{(n)}/D_m^{(n)}$ denote the mth convergent of the continued fraction obtained by omitting the first n partial fractions in (4). Then obviously

$$D_{n+m} = D_m^{(n)} D_n + b_{n+1} x N_m^{(n)} D_{n-1}, \tag{13}$$

which may be written in the form

$$\frac{D_{n+m}}{b_{n+1} x D_n N_m^{(n)}} = \frac{D_m^{(n)}}{b_{n+1} x N_m^{(n)}} + \frac{D_{n-1}}{D_n}. \tag{14}$$

Suppose now that n and m are even integers. By the preceding theorems the zeros and infinities of each fraction in the right hand member of (14) are real and separate each other. But when this is the case, it follows from a well known theorem that the fraction either always increases or always decreases when x describes the real axis in a positive direction.

To ascertain whether there is an increase or a decrease of our two fractions, let m and n be replaced by $2n$ and $2m$ and consider then the signs of the derivatives of the fractions:

$$\frac{D_{2n} D'_{2n-1} - D'_{2n} D_{2n-1}}{(D_{2n})^2}, \quad \frac{b_{2n+1} x N_{2m}^{(2n)} \dfrac{dD_{2m}^{(2n)}}{dx} - b_{2n+1} N_{2m}^{(2n)} D_{2m}^{(2n)} - b_{2n+1} x D_{2m}^{(2n)} \dfrac{dN_{2m}^{(2n)}}{dx}}{(b_{2n+1} x N_{2m}^{(2n)})^2}.$$

For $x = 0$ the numerator of the first fraction reduces to $-b_{2n}$ (see equations (6) and (7)) and that of the second fraction to $-b_{2n+1}$. Since by hypothesis b_{2n} and b_{2n+1} have like signs, the two fractions will increase with increasing x if b_{2n} is negative or decrease if b_{2n} is positive. *A fortiori* the left hand member of (14) will be an increasing or a decreasing function respectively for the entire length of the real axis. Now we know that the zeros and infinities of the left hand member are real. It follows that the zeros and infinities must alternate with one another, or, in other words, that the roots of D_{2n+2m} and $xD_{2n}N_{2m}^{(2n)}$ separate each other.

A parallel course of reasoning applies to prove the remaining conclusions stated in the following theorem:

THEOREM VI.—*The roots of D_{2n+2m} are separated by those of $xD_{2n}N_{2m}^{(2n)}$ and by those of $D_{2n-1}D_{2m}^{(2n)}$. So also the roots of $D_{2n+2m+1}$ alternate with those of $xD_{2n}N_{2m+1}^{(2n)}$ and with those of $D_{2n-1}D_{2m+1}^{(2n)}$.**

It should be pointed out that this theorem is not to be so interpreted as to preclude the possibility of the coincidence of two adjacent roots of the pair of functions compared. As the b_i vary in value, it may happen that two such

* STIELTJES (§5) demonstrates this theorem for his continued fraction in another manner, but the proof just given is simpler.

roots momentarily unite. Suppose, for example, a root of $N_{2m}^{(2n)}$ to coincide with one of D_{2n}. Then it must also be a root of D_{2n+2m}, inasmuch as

$$D_{2n+2m} = D_{2m}^{2n} D_{2n} + b_{2n+1} x N_{2m}^{(2n)} D_{2n-1}. \tag{15}$$

Since, moreover, it is impossible for a root of D_{2n} to coincide with one of D_{2n-1} or a root of $D_{2m}^{(2n)}$ with one of $N_{2m}^{(2n)}$, this is the only case in which a root of D_{2n+2m} unites with a root of $xD_{2n}N_{2m}^{(2n)}$.

Equation (15) shows that when this coincidence takes place, the root of D_{2n+2m} is entirely independent of the value of b_{2n+1}. A like independence exists when a root coincides with a root of $D_{2m}^{(2n)}$ and of D_{2n-1}, and in no other case.

5. *On the condensation of the roots.*—We proceed next to consider the possibility of the accumulation of the roots of D_n in the vicinity of a point of the real axis when n increases indefinitely. Let P be any given point except the origin, and suppose first that the number of roots of D_n in the vicinity of P has no upper limit. Theorem VI shows that *between any two consecutive roots of D_n on the same side of the origin there must always lie at least one root of D_{n+m}*. Our hypothesis therefore leads at once to the conclusion that the number of roots of D_n within a prescribed distance of P will become and remain greater than any prescribed integer, if n be sufficiently increased.

Suppose, on the other hand, that the number of roots of D_n in the vicinity of P has an upper limit l. Then within any sufficiently small interval which contains P as an interior point, there will be exactly l roots of some D_n, while each succeeding D_{n+m} will have either l or $l-1$ roots in the interval. Choose now for consideration such a group of l roots. If they lie on both sides of P, let γ', γ'' denote the two roots adjacent to it. By increasing n another group of l roots of some succeeding D_{n+m} will be obtained which lie within (γ', γ''). But if $l > 2$, this will leave two roots of the former group which include between them no root of the latter, which is impossible. On the other hand, if all the roots of the first group lie on the same side of P, a second group of l roots will ultimately be included between P and the nearest root of the first group. This will be impossible unless $l = 1$. We conclude therefore that when n increases indefinitely, either an unlimited number of roots of D_n will accumulate in the vicinity of P or the number will be equal at most to 2; and if it is equal to 2, the two roots which approach P must lie on opposite sides of this point.

The same reasoning is equally applicable to the accumulation of positive (negative) roots of D_n in the vicinity of the origin. Either the number of roots thus accumulating will be infinite or it will not exceed 1. We shall supplement this result presently by showing that if $b_{2n-1} + b_{2n}$ has no lower limit, an infinite number of positive roots will condense in the vicinity of the origin.

6. *On the limiting positions of the roots.*—Let the roots of D_{2n} now be numbered on each side of the origin in the order in which they are met in proceeding out from this point, those on the positive side being denoted by $x_1', x_2', \cdots$ and those on the opposite side by $x_1'', x_2'', \cdots$. From theorem II it is clear that as n increases, x_i' and x_i'' move always toward the origin. They have, then, definite limiting positions, but the limiting positions of x_i' (or x_i'') for different values of i are not necessarily distinct. It follows, however, from § 5 that if two roots, x_i' and x_{i+1}', have a common limit, this must also be the limit of all the succeeding roots.

When there is an upper limit to the number of roots of D_n on either side of the origin, as in the case considered by STIELTJES, analogous conclusions can be drawn for the roots of D_{2n+1}. Suppose, for example, that the number of roots on the negative side is limited. When once the number of negative roots of D_{2n-1} reaches its maximum, it must remain equal to this maximum for all succeeding values of n because the number of negative roots of D_{2n+1}, as of D_{2n}, is equal to the number of positive signs in the sequence (10). Now the roots of D_{2n+1} are not only separated by those of D_{2n+3} but are included between the extreme roots of the latter. As soon, therefore, as the number of negative roots has attained its maximum, each negative root $\bar{x}_i''$ with a further increase of n must recede from the origin while the ith positive root $\bar{x}_i'$ will approach it. Hence there is a limiting position for the ith root on either side of the origin. If the limits of two consecutive positive roots coincide, all the succeeding positive roots have the same limit, but the negative roots, being restricted in number, must have distinct limits, since two consecutive roots of D_n on the same side of the origin will include between them at least one root of each D_{n-m}.

When the roots of D_{2n} and D_{2n+1} are enumerated in the opposite order—that is, in the order in which they occur in going from the point at infinity toward the origin—we shall denote those on the positive side by y_i' and $\bar{y}_i'$ respectively and those on the opposite side by y_i'' and $\bar{y}_i''$. Since the roots of D_{2n+3} separate and enclose those of D_{2n+1}, the ith roots $\bar{y}_i', \bar{y}_i''$ move with increasing n toward the point at infinity, and each has a limiting position. Corresponding conclusions can be made for the roots of D_{2n} if its positive or negative roots are restricted in number.

7. *Conditions for the indefinite approach of roots to the origin.* The important question now naturally arises: what is the necessary and sufficient condition that the roots of D_n with increasing n shall approach infinitesimally near to the origin? This question was examined by STIELTJES* for continued fractions with positive coefficients, and he found the condition to be that b_n should have no upper limit. Further, I have shown† that if $|b_n|$ has an upper limit L in

* Loc. cit., §§ 9, 10.

† Transactions of the American Mathematical Society, vol. 2 (1901), p. 477, § 2.

any continued fraction whatsoever of the form (4), whether with real or complex coefficients, the numerators and denominators of the convergents can not vanish in the interior of a circle having its center in the origin and a radius equal to $1/4L$. Therefore a necessary condition for the approach of roots to within less than any assignable distance of the origin is that $|b_n|$ shall have no upper limit. We shall consider now the question of the sufficiency of this condition for the class of continued fractions treated in this paper, adhering here as far as possible to the methods of STIELTJES.

Let us take first the hypothesis that $|b_{2n+1}|$ has no upper limit. By a previous theorem D_{2n} and D_{2n+1} have on either side of the origin the same number of roots. Also from (6) and (7)

$$\Sigma\left(\frac{1}{\bar{x}_i'}\right)+\Sigma\left(\frac{1}{\bar{x}_i''}\right)=-(b_1+b_2+\cdots+b_{2n+1}),$$

$$\Sigma\left(\frac{1}{x_i'}\right)+\Sigma\left(\frac{1}{x_i''}\right)=-(b_1+b_2+\cdots+b_{2n}):$$

whence it follows that

$$\Sigma\left(\frac{1}{\bar{x}_i'}-\frac{1}{x_i'}\right)+\Sigma\left(\frac{1}{x_i''}-\frac{1}{\bar{x}_i''}\right)=-b_{2n+1}.$$

If b_{2n+1} is negative, let the last equation be written in the form

$$\frac{1}{\bar{x}_1'}=-b_{2n+1}+\Sigma\left(\frac{1}{x_i''}-\frac{1}{\bar{x}_i''}\right)+\sum_{i=1}^{k-1}\left(\frac{1}{\bar{x}_i'}-\frac{1}{x_{i+1}'}\right)+\frac{1}{x_k'} \tag{16}$$

in which x_k' denotes the largest positive root of D_{2n}. Each of the binomial terms on the right hand side is positive because of the mode of alternation of the roots of D_{2n} and D_{2n+1} (theorem III). It follows that $1/\bar{x}_1' > -b_{2n+1}$. Consequently, if there is a set of negative coefficients b_{2n+1} whose absolute values have no upper limits, the smallest positive root of D_{2n+1} will approach within less than any assigned distance of the origin. In exactly the same manner it can be shown that the first negative root of D_{2n+1} will come indefinitely near to the origin if the positive coefficients b_{2n+1} have no upper limit.

Suppose next that $|b_{2n+2}|$ has no upper limit while $|b_{2n+1}|$ has. If the roots of D_{2n+2} on the two sides of the origin are denoted in order by v_i', v_i'', one and only one of these roots—call it v_k—lies further away from the origin than any root of D_{2n} on the same side. To every other root v there corresponds one of D_{2n}, and the equation analogous to (16) which connects the two systems of roots is

$$\Sigma\left(\frac{1}{v_i'}-\frac{1}{x_i'}\right)+\Sigma\left(\frac{1}{v_i''}-\frac{1}{x_i''}\right)+\frac{1}{v_k}=-(b_{2n+1}+b_{2n+2}).$$

By theorem I the sign of v_k will be opposite to that of b_{2n+2}. If, then, there is any set of negative coefficients b_{2n+2} whose absolute values have no upper limit, we shall rearrange the terms of the last equation as follows:

$$\frac{1}{v_1'} = -(b_{2n+1} + b_{2n+2}) + \sum_{i=1}^{k-2}\left(\frac{1}{x_i'} - \frac{1}{v_{i+1}'}\right) + \Sigma\left(\frac{1}{x_i''} - \frac{1}{v_i''}\right) + \left(\frac{1}{x_{k-1}'} - \frac{1}{v_k}\right).$$

By theorem II all the binomial terms after the first will be positive; hence

$$\frac{1}{v_1'} > -(b_{2n+1} + b_{2n+2}).$$

But by hypothesis there is some set of values of n for which the right hand member of this inequality is positive and has no upper limit. It follows that v_1' must come within any assigned distance of the origin, and as it diminishes with n, this point must be its limit. The same conclusion will hold also for the first positive root of D_{2n+3} since the root lies between the origin and v_1'. Parallel reasoning shows that the first negative root of D_{2n} and of D_{2n+1} has the origin as a limit whenever there is any set of values for which $b_{2n+1} + b_{2n+2}$ is positive and has no upper limit.

It has thus been established that if $|b_n|$ has no upper limit, either the first positive or first negative root of D_{2n+1}, or both, will come indefinitely near to the origin, though this point need not always be a limit. The suppositions made in the two preceding paragraphs concerning the even and odd elements b_{2n}, b_{2n+1} are obviously not affected by the omission of an *even* number of partial fractions at the outset of (4). The same conclusions therefore are valid for $D_{2m+1}^{(2n)}$ and $D_{2m}^{(2n)}$ as for D_{2n+1} and D_{2n+2}. Accordingly, if there is any set of values of n for which $b_{2n-1} + b_{2n}$ is negative (positive) and has no upper limit, the first positive (negative) roots of D_{2n-1} and $D_{2m+1}^{(2n)}$ will not only come within any assigned distance of the origin but will remain there if m and n are sufficiently great.

Suppose then that there is such a set of values for which $b_{2n-1} + b_{2n}$ is negative. The same then is true either of b_{2n} or of b_{2n-1}; and if true of the latter, it is true also of the former, since b_{2n-1} and b_{2n-2} have like signs. We may therefore assume that there is an infinite sub-set of values for which b_{2n} also is negative.

By theorem VI the $n + m$ roots of $D_{2n+2m+1}$ alternate with $n + m - 1$ roots of $D_{2n-1}D_{2m+1}^{(2n)}$. Hence on one side of the origin or the other the nearest root will belong to $D_{2n+2m+1}$, and this side will be the one on which there lie more roots of this polynomial than of $D_{2n-1}D_{2m+1}^{(2n)}$. Now the number of positive roots of D_{2n-1} and of $D_{2n+2m+1}$ is equal to the number of negative signs occurring in the first $n - 1$ and $n + m$ terms of the sequence

$$b_2, b_4, b_6, \cdots,$$

while the number of positive roots of D_{2m+1}^{2n} is equal to the number of negative signs in the first n terms of the sequence subsequent to b_{2n}. Consequently, when b_{2n} is negative, the first positive root of $D_{2n+2m+1}$ lies nearer to the origin than the first positive root of $D_{2n-1}D_{2m+1}^{2n}$. Since also the two systems of roots alternate, we conclude that *two* positive roots of $D_{2n+2m+1}$ must come and remain within any assigned distance of the origin. This conclusion can now be transferred to D_{2n-1} and D_{2m+1}^{2n} if n and m are sufficiently great. By repeating then the reasoning we next obtain three positive roots of $D_{2n+2m+1}$ within the assigned distance of the origin, and so on. As also its roots are separated by those of D_{2n+2m}, we see finally that the origin must be a point of condensation * of an infinite number of positive roots of D_n, if there is any set of values for which $b_{2n-1}+b_{2n}$ is negative and has numerically no upper limit.

The result thus reached may be recapitulated as follows:

THEOREM VII.—*The necessary and sufficient condition that the distance between the origin and the nearest root of D_n shall have no lower limit is that $|b_n|$ shall have no upper limit. If there is any set of values of n for which $b_{2n-1}+b_{2n}$ has numerically no upper limit and is negative (positive), an infinitely increasing number of positive (negative) roots of D_n will accumulate with increasing n in the vicinity of the origin.*

8. *The special case in which $\lim b_n = 0$.*—There seems to be no simple test for the indefinite increase of the outermost root or roots of D_n. A sufficient condition, however, is that $|b_{2n}|$ shall have no lower limit. This can be inferred from the fact that $-b_{2n}$ is the quotient of the product of the roots of D_{2n-2} by those of D_{2n}. For since the roots x_i', x_i'' of D_{2n} lie nearer to the origin than the corresponding roots of D_{2n-2} and since the former polynomial has one more root than the latter, the distance of this extra root from the origin must for some value of n exceed any assigned quantity if $|b_{2n}|$ has no lower limit.

A special and important case which I shall examine is that in which b_n has the limit 0. In a preceding paper † I have given a very simple proof that when $\lim b_n = 0$ in *any* continued fraction of the form (4), the continued fraction represents a meromorphic function and converges in all the points of the x-plane with the exception of the poles of this function. As n increases, the roots of D_n which lie within any given finite distance of the origin approach limiting positions, and these positions are the poles of the function.

I shall supplement this result for the continued fractions of the present paper by showing, conversely, when it represents a meromorphic function, $\lim b_n = 0$. In proving this we may start with the hypothesis, apparently less restrictive, that the roots $\bar{x}_i'$, $\bar{x}_i''$ of D_{2n+1} with increasing n approach limits and that these

* The reasoning by which the existence of the point of condensation is established could be applied to the continued fraction of STIELTJES but is not used by him.

† Transactions of the American Mathematical Society, vol. 2 (1901), p. 476.

limits are distinct from one another and from 0. Let the roots of D_{2n-1} be denoted by $\bar{v}_i'$, $\bar{v}_i''$. These roots alternate with those of D_{2n+1}, and on one side of the origin there is one more root of the latter polynomial than of the former. Let $\bar{x}_k$ be the outermost root on this side. Then we have

$$(17) \qquad \Sigma\left(\frac{1}{\bar{x}_i'}-\frac{1}{\bar{v}_i'}\right)+\Sigma\left(\frac{1}{\bar{x}_i''}-\frac{1}{\bar{v}_i''}\right)+\frac{1}{\bar{x}_k}=-(b_{2n}+b_{2n+1}).$$

Choose now any interval $(s, -s)$ of the axis of x so large that $1/s$ shall be less than any arbitrarily prescribed quantity $\epsilon/4$. In consequence of the assumption of limiting positions for the roots the differences $|1/\bar{x}_i' - 1/\bar{v}_i'|$ and $|1/\bar{x}_i - 1/\bar{v}_i''|$ can be made as small as we wish for all pairs of roots which lie within the segment, and as the number of such pairs is finite, the sum of these differences can be made less then $\epsilon/4$. Consider next the positive roots outside of the interval. We have

$$\left|\frac{1}{\bar{x}_j'}-\frac{1}{\bar{v}_j'}+\frac{1}{\bar{x}_{j+1}'}-\frac{1}{\bar{v}_{j+1}'}+\cdots-\frac{1}{\bar{v}_{j+s}'}\right|<\frac{1}{|\bar{x}_j'|}<\epsilon/4$$

since the terms are alternately positive and negative and each less than the preceding owing to the alternation of the roots. A similar inequality holds for the negative roots exterior to the interval. Lastly, if n is sufficiently large, $1/|x_k|<\epsilon/4$. It follows therefore from (17) that we may make

$$|b_{2n}+b_{2n+1}|<\epsilon.$$

Hence, as b_{2n} and b_{2n-1} have like signs, $\lim b_n = 0$. But this is what we wished to prove.

A comparison between the roots of D_{2n-1} and D_{2n} next gives

$$\left|\Sigma\left(\frac{1}{x_i'}-\frac{1}{\bar{v}_i'}\right)+\Sigma\left(\frac{1}{x_i''}-\frac{1}{\bar{v}_i''}\right)+\frac{1}{x_k}\right| = |-b_{2n}|<\epsilon.$$

The binomial terms here have all the same sign as $1/x_k$ because of the alternation of the roots of the two polynomials. Consequently we have

$$\left|\frac{1}{x_i'}-\frac{1}{\bar{v}_i'}\right|<\epsilon, \qquad \left|\frac{1}{x_i''}-\frac{1}{\bar{v}_i''}\right|<\epsilon,$$

and the limiting positions of the roots of D_{2n} therefore are identical with those of D_{2n-1}.

From the identity of these positions the convergence of the continued fraction can be argued. The manner in which this is done is indicated in § 14.

II. Derivation of the integral $\int_a^b \frac{d\Phi(u)}{z+u}$.

9. *Change of variable.*—It will be convenient now to make the substitution $x = 1/z$. Our continued fraction (4), after the removal of a factor z, can then be thrown into the form

$$\cfrac{1}{z+\cfrac{b_1}{1+\cfrac{b_2}{z+\cfrac{b_3}{1+\cfrac{b_4}{z+\cdots}}}}}. \tag{18}$$

If its convergents are denoted by $N_{2n}(z)/D_{2n}(z)$ and $N_{2n+1}(z)/zD_{2n+1}(z)$, we have

$$D_n(z) = z^{n_1} D_n(x), \qquad N_n = z^{n_2} N_n(x),$$

in which n_1 and n_2 denote the degrees of $D_n(x)$ and $N_n(x)$. It should be noticed that the coefficients of the highest powers of z in $D_n(z)$ and $N_n(z)$ are equal to 1. No confusion will result if hereafter we abbreviate these symbols by D_n and N_n, as the context will show whether the argument z or x is intended.

The modifications to be made in the theorems of Section I owing to the exchange of the origin and point at infinity in the transformation $z = 1/x$ are too simple to require statement. We merely call attention to the fact that the roots of D_{2n} and N_{2n}, also of zD_{2n+1} and N_{2n+1} will alternate along the entire axis of z. Decomposition into partial fractions therefore gives

$$\frac{N_{2n}}{D_{2n}} = \frac{M_1}{z+z_1} + \frac{M_2}{z+z_2} + \cdots + \frac{M_n}{z+z_n} \qquad (z_1 < z_2 < \cdots < z_n), \tag{19}$$

$$\frac{N_{2n+1}}{zD_{2n+1}} = \frac{N_1}{z+z_1'} + \frac{N_2}{z+z_2'} + \cdots + \frac{N_0}{z} + \cdots + \frac{N_n}{z+z_n'} \tag{20}$$

$$(z_1' < z_2' < \cdots < 0 < \cdots < z_n'),$$

in which M_i *and* N_i *are positive* and

$$1 = M_1 + M_2 + \cdots + M_n = N_1 + N_2 + \cdots + N_0 + \cdots + N_n. \tag{21}$$

10. *The function* $\phi_n(u)$.—We now shall consider the even and the odd convergents separately and, after the precedent of Stieltjes, construct for every even convergent a monotone function ϕ_n of a real variable u in the following manner:

$$\begin{aligned}
&\phi_n(u) = 0, && -\infty < u < z_1;\\
&\phi_n(u) = M_1, && z_1 \leqq u < z_2;\\
&\phi_n(u) = M_1 + M_2, && z_2 \leqq u < z_3;\\
&\cdots\cdots\cdots && \cdots\cdots\\
&\phi_n(u) = M_1 + M_2 + \cdots + M_{n-1}, && z_{n-1} \leqq u < z_n;\\
&\phi_n(u) = M_1 + M_2 + \cdots + M_n, && z_n \leqq u < +\infty.
\end{aligned}$$

The relation which subsists between any two functions thus constructed is fundamental for the formation of an integral which shall represent the limit of the continued fraction, when convergent. Since $\phi_{n+m}(u)$ and $\phi_n(u)$ are both increasing functions, their difference can change sign but once between two consecutive roots of $D_n(-z)$. After the last root z_n such a change is impossible because $\phi_{n+m}(u) \leqq \sum M_i = \phi_n(z_n)$. Hence the maximum number of changes of sign is $2n-1$. This number will be reached if, and only if, $\phi_{n+m}(z) - \phi_n(z)$ is negative for each of the n roots z_i of $D_n(-z)$ and is positive for a (smaller) value of z just short of z_i. We will show this to be the case.*

From (14) and the analogous equation connecting $N_{n+m}(x)$ and $N_n(x)$ there results the equation

$$\frac{1}{b_{2n+1}x}\left(\frac{N_{2n+2m}}{N_{2n}N_{2m}^{(2n)}} - \frac{D_{2n+2m}}{D_{2n}N_{2m}^{(2n)}}\right) = \frac{N_{2n-1}}{N_{2n}} - \frac{D_{2n-1}}{D_{2n}},$$

or

$$\frac{N_{2n+2m}}{D_{2n+2m}} - \frac{N_{2n}}{D_{2n}} = \frac{b_{2n+1}xN_{2m}^{(2n)}(D_{2n}N_{2n-1} - N_{2n}D_{2n})}{D_{2n}D_{2n+2m}} = \frac{b_2b_3\cdots b_{2n+1}x^{2n}N_{2m}^{(2n)}}{D_{2n}D_{2n+2m}}.$$

Hence, changing to the variable z, we obtain

$$\left(\frac{M_1^{(n+m)}}{z+z_1^{(n+m)}} + \cdots + \frac{M_{n+m}^{(n+m)}}{z+z_{n+m}^{(n+m)}}\right) - \left(\frac{M_1}{z+z_1} + \cdots + \frac{M_n}{z+z_n}\right) = \frac{b_2b_3\cdots b_{2n+1}\Pi(z+w_i)}{\Pi(z+z_i^{(n+m)})\Pi(z+z_i)}, \tag{22}$$

in which $z_i^{(n+m)}$ and w_i denote in increasing order of magnitude the roots of $D_{2n+2m}(-z)$ and $N_{2m}^{(2n)}(-z)$ respectively.

Without altering in any manner the form of (22) the roots z_i, $z_i^{(n+m)}$, and w_i can be brought by a change of origin into the positive half of the axis of z. We will suppose for the moment that this has been done.

By theorem VI the $n+m-1$ roots z_i, w_i alternate with the $n+m$ roots of $D_{2n+2m}(-z)$, and any particular root z_j of $D_{2n}(-z)$ must be followed and preceded immediately by a root of $D_{2n+2m}(-z)$. Let $z_k^{(n+m)}$ denote the succeeding root. Then within a circular ring having the origin as a center and the inner and outer radii

$$R = z_j, \qquad R' = z_k^{(n+m)},$$

the two members of (22) can be expanded into a LAURENT's power series $P_1(z) + P_2(1/z)$. The coefficient of $1/z$ on the left hand side of the equation is obviously

*STIELTJES establishes the fact with the aid of certain integrals of a simple nature (loc. cit., § 42). The proof with slight modifications is also applicable here, but I have preferred to give a new proof based directly upon the relation between the convergents.

$$\sum_{i=1}^{k-1} M_i^{(n+m)} - \sum_{i=1}^{j} M_i = \phi_{n+m}(z_j) - \phi_n(z_j),$$

which is exactly the expression whose sign is desired. We shall determine its sign by seeking the coefficient of $1/z$ on the right hand side. There the positive product $b_2 b_3 \cdots b_{2n+1}$ can be neglected. Replace momentarily z by $-z$. If any factor of the denominator is denoted by $z - z'$, its expansion into a series gives

$$\frac{1}{z - z'} = -\frac{1}{z'}\left(1 + \frac{z}{z'} + \frac{z^2}{(z')^2} + \cdots\right) \tag{23}$$

or

$$\frac{1}{z - z'} = \frac{1}{z}\left(1 + \frac{z'}{z} + \frac{(z')^2}{z^2} + \cdots\right) \tag{24}$$

according as $z' \geqq R'$ or $z' \leqq R$. All the terms within the parentheses have positive coefficients since z' has been brought to the positive side of the origin. The corresponding expansions for the factors of the numerator on the right hand side of (22) are

$$-w'\left(1 - \frac{z}{w'}\right) \qquad (w' > R'), \tag{25}$$

$$z\left(1 - \frac{w'}{z}\right) \qquad (w' < R). \tag{26}$$

Let z'' be the root of $D_{2n+2m}(-z)$ which immediately precedes any assigned root w' of $N_{2m}^{(2n)}(-z)$ and z''' the root which immediately follows it. If $w' > R'$, multiply (25) into the expansion (23) for which $z' = z''$, and if $w' < R$, multiply (26) into the expansion (24) for which $z' = z'''$. Since $z'' < w' < z'''$, the product of the two parentheses in either case will be a series with positive coefficients, and the continued multiplication of all such products gives a LAURENT's power series, the coefficients of which are positive. Consequently we need to attend only to the product of the constant terms without the parentheses. Now these terms, $-1/z'$ and $-w'$, are negative and correspond to roots of $N_{2m}^{(2n)}(-z)$, $D_{2n}(-z)$, and $D_{2n+2m}(-z)$ which are equal to or greater than R'. As the total number of such roots is odd in consequence of the alternation of the roots of D_{2n+2m} and $D_{2n}N_{2m}^{(2n)}$, we come finally to the conclusion that the coefficient of $1/z$ in the Laurentian expansion of the right hand member of (22) must be negative. In other words $\phi_{n+m}(z_i) - \phi_n(z_i)$ is negative, as was to be proved.

If a LAURENT's power series for (22) is formed for values of z slightly less than z_j, it will be valid for a certain circular ring whose outer radius is $R = |z_j|$, and the number of expansions of the form (23) and (25) will be less by 1 than before. Hence $\phi_{n+m}(z) - \phi_n(z)$ will positive. This completes the

proof that $\phi_{n+m}(u) - \phi_n(u)$ changes sign the maximum number of times, once at each root of D_{2n} (except the last) and once between any two of its roots which are consecutive.

11. *The function* $\psi_n(u)$.—For each odd convergent N_{2n+1}/zD_{2n+1} a monotone function $\psi_n(u)$ can be similarly constructed having its discontinuities in the roots z_i' of $D_{2n+1}(z)$. Thus

$$\begin{aligned}
&\psi_n(u) = 0, && -\infty < u < z_1';\\
&\psi_n(u) = N_1, && z_1' \leqq u < z_2';\\
&\psi_n(u) = N_1 + N_2, && z_2' \leqq u < z_3';\\
&\quad . \quad . \quad . \quad . \quad . && \quad . \quad . \quad .
\end{aligned}$$

To compare $\psi_{n+m}(u)$ and $\psi_n(u)$ use may be made of the relation

$$\frac{N_{2n+2m+1}}{zD_{2n+2m+1}} - \frac{N_{2n+1}}{zD_{2n+1}} = -\frac{b_2 b_3 \cdots b_{2n+2} N_{2m}^{(2n+1)}}{zD_{2n+1} D_{2n+2m+1}}.$$

Since by theorem VI the roots of $D_{2n+2m+1}(z)$ alternate with those of

$$zD_{2n+1}(z) D_{2m-1}^{(2n+2)}(z) \equiv zD_{2n+1}(z) N_{2m}^{(2n+1)}(z),$$

it follows by reasoning like that of § 10 that $\psi_{n+m}(z_i') - \psi_n(z_i')$ has a constant sign for all roots of D_{2n+1} and changes sign when u passes through each root excepting either the first or the last. Since also $\psi_{n+m}(u)$ and $\psi_n(u)$ are increasing functions, $\psi_{n+m}(z_i') - \psi_n(z_i')$ must be negative.

We may add that $\psi_{n+m}(z_i) - \phi_n(z_i)$ and $\phi_{n+m}(z_i') - \psi_n(z_i')$ are both negative.

12. *First special case.*—The relations established in §§ 10–11 are insufficient in themselves to prove the convergence of the continued fraction or of either sequence of alternate convergents. STIELTJES, for his continued fraction, first demonstrated the convergence of the two sequences and then used the above relations to prove that their limits could be expressed as integrals of the form (3). This course can not be followed here since the two sequences are not always convergent, as will appear later.* Consequently in place of a general discussion we will take up certain cases wherein such supplementary conditions are imposed as to ensure simultaneously both the convergence of the sequence and the existence of the integral form for its limit.

*In deriving the integral (3) STIELTJES employs also a certain accessory condition $\psi(a) \leqq \chi(b)$ (loc. cit., §§ 41, 43). The proof of this condition assumes not only the convergence of the sequence considered but also a notable theorem concerning a LAURENT's power series (§§ 31–36). The latter can not always be applied to the continued fractions of this paper.

The two most diverse suppositions which may be entertained are (1) that the minimum distance between two consecutive roots of D_n as n increases shall have a finite lower limit and (2) that the maximum distance between two consecutive roots in any arbitrary finite portion of the axis shall diminish indefinitely when n is increased.

We shall take first the latter hypothesis and prove the convergence of the continued fraction at any assigned point z exterior to the real axis. Consider first the even convergents. To prove their convergence it is necessary to prove that $|N_{2n}/D_{2n} - N_{2n+2m}/D_{2n+2m}|$ can be made less than any prescribed quantity ϵ by sufficiently increasing n. Now N_{2n}/D_{2n} consists of the sum of n terms $M_i/(z+z_i)$ while N_{2n+2m}/D_{2n+2m} comprises a greater number of such terms. For comparison the latter may be divided into n groups in the following manner. We know that between any two successive roots z_i and z_{i+1}, of $D_{2n}(-z)$, there is a root of $D_{2n+2m}(-z)$—call it $z_{(i)}^{(n+m)}$—at which $\phi_{n+m}(u)$ receives an increment which first makes it equal to or greater than $\phi_n(z_i)$. This increment may be divided into two parts $M'_{(i)}$ and $M''_{(i)}$, the former of which is an increment just sufficient to make $\phi_{n+m}(u)$ equal to $\phi_n(z_i)$. The first group of terms is to consist of

$$\frac{M_1^{(n+m)}}{z+z_1^{(n+m)}} + \cdots + \frac{M_{(1)-1}^{(n+m)}}{z+z_{(1)-1}^{(n+m)}} + \frac{M'_{(1)}}{z+z_{(1)}^{(n+m)}} \qquad (M_1^{(n+m)} + \cdots + M'_{(1)} = M_1),$$

the second group of

$$\frac{M''_{(1)}}{z+z_{(1)}^{(n+m)}} + \frac{M_{(1)+1}^{(n+m)}}{z+z_{(1)+1}^{(n+m)}} + \cdots + \frac{M'_{(2)}}{z+z_{(2)}^{(n+m)}} \qquad (M''_{(1)} + \cdots + M'_{(2)} = M_2),$$

and so on. Then it is obvious that the difference between the ith term of N_{2n}/D_{2n} and the ith group will have an absolute value not exceeding the larger of the two quantities

$$(27) \qquad M_i \left| \frac{1}{z+z_i} - \frac{1}{z+z_{i-1}} \right| = \frac{M_i |z_i - z_{i-1}|}{|z+z_i| \cdot |z+z_{i-1}|}, \qquad M_i \left| \frac{1}{z+z_i} - \frac{1}{z+z_{i+1}} \right|.$$

If $i=1$, $M_i/|1(z+z_1) - 1/(z+z_1^{(n+m)}|$ is to be substituted for the former of these expressions, and if $i=n$, a similar expression is to be substituted for the latter.

Take now any arbitrary segment of the real axis. The n differences (27) can be divided into two sets comprising the differences which correspond respectively to roots of D_{2n} which lie within the segment and to those which lie without. Let δ_i be the greater of the distances between any root z_i within the segment and either adjacent root * of D_{2n} (whether in or out of the segment), and let ρ denote the distance of any point of the complex plane from the real axis.

* If $i=1$, we use the greater of the distances $|z_1 - z_1^{(n+m)}|$ and $|z_1 - z_2|$.

Then the larger of the differences (27) will be less than $M_i\delta_i/\rho^2$ and their sum less than $\sum M_i\delta_i/\rho^2$, the summation applying only to roots within the segment. But by virtue of our hypothesis the maximum of the distances δ_i may be indefinitely diminished by increasing n sufficiently. Since also $\sum M_i$ is limited in value, it follows that we may make $\sum M_i\delta_i/\rho^2 < \epsilon/2$.

It remains to consider the second set of differences. By extending the segment sufficiently in both directions, $1/|z+z_i|$ may be made as small as we choose for all roots which lie without the segment. Since also the distance between the first root without the segment and the first root within can be indefinitely diminished by increasing n, we can make $|1/(z+z_i)-1/(z+z_{i+1})|$ as small as is desired for each root z_i without the segment, inclusive of the first root. Then since $\sum M_i$ is limited it follows that the sum of the second set of differences will also become less than $\epsilon/2$, and consequently

$$|N_{2n}/D_{2n} - N_{2n+m}/D_{2n+2m}| < \epsilon/2 + \epsilon/2 = \epsilon.$$

But this is what we had to prove.

In an exactly similar manner it can be shown that

$$|N_{2n}/D_{2n} - N_{2n+2m+1}/D_{2n+2m+1}| < \epsilon.$$

We conclude therefore that the continued fraction is convergent at any point not on the real axis. Furthermore, if any closed region T is taken which lies entirely in the interior of the positive or the negative half plane, ρ will have for this region a lower limit. The convergence is therefore uniform, and the limit of the continued fraction is accordingly holomorphic within T.

To put the limit into the form of an integral, we shall proceed now in the same manner as STIELTJES* and construct next a new function $\Phi(u)$ as follows. If u is for the moment taken to be a fixed point of the real axis and n assumes all values greater than a given integer, $\phi_n(u)$ will have both an upper and a lower limit. As the given integer is increased, the former limit can only decrease and the latter can only increase. Let $\psi(u)$ and $\chi(u)$ be the limits to which they approach, and put

$$\Phi(u) = \frac{\chi(u)+\psi(u)}{2}.$$

It is apparent that $\chi(u)$, $\psi(u)$, and $\Phi(u)$ are all increasing functions of u since $\phi_n(u)$ increases with u. If c and d be the upper and lower limits to the roots of $D_{2n}(z)$, put

$$F(z) = \int_c^d \frac{d\Phi(u)}{z+u} \qquad (-\infty \leqq c < d \leqq +\infty).$$

Then since

* Loc. cit., §§ 41, 44.

$$\frac{N_{2n}(z)}{D_{2n}(z)} = \int_{+\infty}^{-\infty} \frac{d\phi_n(u)}{z+u},$$

the function $F(z)$ bears exactly the same relation to $\Phi(u)$ as $N_{2n}(z)/D_{2n}(z)$ to $\phi_n(u)$. Moreover, as $\phi_{n+m}(z_i) - \phi_n(z_i) \prec 0$, it follows that

$$\Phi(z_i) - \phi_n(z_i) \leqq 0.$$

On the other hand, when $u = z_i - h$ (h infinitesimal and positive),

$$\phi_{n+m}(u) - \phi_n(u) \succ 0$$

and hence $\Phi(u) - \phi_n(u) \geqq 0$. The difference $\Phi(u) - \phi_n(u)$ has accordingly properties similar to those of $\phi_{n+m}(u) - \phi_n(u)$ which we may use in the same manner to prove that we can make

$$|F(z) - N_{2n}/D_{2n}| \prec \epsilon$$

by increasing n sufficiently; *$F(z)$ is therefore the limit of the continued fraction.*

13. *Second special case.*— We proceed now to the case in which the minimum distance between two consecutive roots of D_n has a lower limit. By § 6 the roots $1/\bar{y}_i'$, $1/\bar{y}_i''$ of $D_{2n+1}(-z)$ move with increasing n toward the origin of the z-plane. Call their limiting values ξ_i', ξ_i''. If any interval $(s, -s)$ of the real axis is taken, there will be a finite number of these values in the interval, and hence D_{2n+1} and $D_{2n+2m+1}$ for a sufficiently great value of n will have precisely the same number of roots in the interval.

Consider those which lie upon the negative side of the origin. Since $1/\bar{y}_i'' \prec \xi_i'' \prec 1/\bar{y}_{i-1}''$, we have $\psi_n(1/\bar{y}_i'') = \psi_n(\xi_i'')$. Hence in the decomposition of N_{2n+1}/zD_{2n+1} and $N_{2n+2m+1}/zD_{2n+2+1}$ into partial fractions the terms which correspond to the negative roots of the denominators may be written as

$$(28) \qquad \frac{\psi_n(\xi_i'') - \psi_n(\xi_{i-1}'')}{z + 1/\bar{y}_i''}, \qquad \frac{\psi_{n+m}(\xi_i'') - \psi_{n+m}(\xi_{i-1}'')}{z + 1/y_i^{(n+m)}}.$$

Now $\psi_{n+m}(\xi_i'') = \psi_{n+m}(y_i^{(n+m)}) \succ \psi_n(\bar{y}_i'') = \psi_n(\xi_i'')$, which shows that $\psi_n(\xi_i'')$ increases with n to a limit. The difference of the numerators of the two fractions (28) can therefore be made as small as desired. If z is not on the real axis, this is also true of their denominators, since their roots have a common limit. The difference between the two fractions will therefore be diminished indefinitely by increasing n. A like result holds for the roots on the positive side of the origin and for the root at the origin. Hence the sum of the differences of the partial fractions for the roots of D_{2n+1} and $D_{2n+2m+1}$ within the interval can be made smaller than $\epsilon/2$.

By enlarging the interval chosen $1/|z+1/y_i|$ can be made arbitrarily small for all roots of the two polynomials which lie without the intervals. Hence the absolute values of the sums of the partial fractions

$$\sum M_i/|z+1/y_i|, \qquad \sum M_i^{(n+m)}/|z+1/y_i^{(n+m)}|$$

corresponding to these roots can be made less than $\epsilon/4$. We have accordingly

$$\left|\frac{N_{2n+1}}{zD_{2n+1}}-\frac{N_{2n+2m+1}}{zD_{2n+2m+1}}\right| < \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{2}=\epsilon,$$

which shows that the sequence of odd convergents converges at any point z not on the real axis.

The proof can be extended to all points on the real axis except ξ_i', ξ_i'', and it is easy to see that the convergence is uniform in any closed region which excludes these points.

The even convergents do not permit of similar treatment.

14. The methods of §§ 12, 13 can be extended to other cases, as for example the one mentioned in § 8. We found there that the roots of D_n had limiting positions separated by finite distances in the plane of x. In the z-plane there will be a point of condensation of these limiting positions at the origin. The principles of § 12 can be applied to an arbitrarily small interval containing the origin as an interior point and the remainder of the axis can be treated as in § 13.

III. On the Convergence of the Continued Fraction and Corresponding Infinite Series.

15. *Character of the series.* — It is well known that to every continued fraction of the form (4) there belongs an infinite series,

$$c_0 - c_1x + c_2x^2 - c_3x^3 + \cdots \qquad (c_0=1) \tag{29}$$

with which the expansion of $N_n(x)/D_n(x)$ in ascending powers of x agrees for a number of terms equal to the sum of the degrees of numerator and denom inator, increased by 1. Not only can the series be obtained from the continued fraction, but conversely, if the series is given, the continued fraction can be found. If, in short, we put *

$$A_n=\begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix}, \quad B_n=\begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix}, \quad A_0=B_0=1,$$

* Stieltjes, loc. cit., § 11. The expressions for the elements of the continued fraction in terms of those of the series go back at least to Frobenius, Crelle's Journal, vol. 90 (1881), pp. 1–5.

and also $$a_{2n} = \frac{A_n^2}{B_{n-1} B_n}, \qquad a_{2n+1} = \frac{B_n^2}{A_n A_{n+1}},$$

we have then for the coefficients of the continued fraction

$$b_n = \frac{1}{a_{n-1} a_n}.$$

To furnish therefore a continued fraction in which b_{2n} and b_{2n+1} have like signs, the series (29) must be subjected to the restrictions

$$A_n > 0, \qquad B_n \neq 0. \tag{30}$$

On the other hand, if b_{2n-1} and b_{2n} are to be of like sign, we have

$$B_n > 0, \qquad A_n \neq 0. \tag{31}$$

The quantities a_n are themselves the coefficients of a continued fraction of the form (1), to which STIELTJES gives the preference. To the conditions (30) and (31) correspond two classes of continued fractions in which a_{2n+1} and a_{2n} respectively are positive.

The awkward secondary conditions $B_n \neq 0$, $A_n \neq 0$ were necessary above to prevent b_n from becoming infinite and the continued fraction in consequence illusory. By reference to (8) it will be seen at once that the even convergents of (4) generate by themselves a continued fraction which, in terms of z, has the familiar form

$$z\left[\frac{\lambda_0}{z + d_1} - \frac{\lambda_1}{z + d_2} - \frac{\lambda_2}{z + d_3} - \cdots\right] \tag{32}$$

in which

$$\lambda_0 = b_0 = 1, \qquad \lambda_n = b_{2n} b_{2n+1} = \frac{A_{n-1} A_{n+1}}{A_n^2},$$

$$d_1 = b_2 = \frac{B_0 B_1}{A_1^2}, \qquad d_{n+1} = b_{2n+1} + b_{2n+2} = \frac{A_{n+1}^2 B_{n-1} + A_n^2 B_{n+1}}{A_n A_{n+1} B_n}.$$

Here the λ_n depend only upon the A_n, and the d_n can not become infinite as long as $A_n \neq 0$, since B_n, which appears in the denominator of d_{n+1}, is contained also as a factor in its numerator.* *The continued fraction* (32) *therefore can not become illusory if* $A_n \neq 0$ *for all values of* n.

* For verification of this statement it is convenient to refer forward to § 20. In the notation of this paragraph $A_n \equiv c_{n-1,\,n}$ or $B_n = c_{n,\,n}$. Now by a familiar theorem of determinants

$$c_{\alpha\beta}^2 = c_{\alpha+1,\,\beta}\, c_{\alpha-1,\,\beta} - c_{\alpha,\,\beta+1}\, c_{\alpha,\,\beta-1}.$$

Hence

$$A_n^2 = B_n c_{n-2,\,n} - c_{n-1,\,n+1} B_{n-1}$$

$$A_{n+1}^2 = B_{n+1} c_{n-1,\,n+1} - c_{n,\,n+2} B_n.$$

The substitution of these values in the numerator of d_{n+1} gives the desired result.

If $B_n \neq 0$, we obtain in similar manner from the odd convergents a continued fraction

$$1 - \frac{\lambda_1'}{z + d_1' -} \frac{\lambda_2'}{z + d_2' -} \cdots \tag{33}$$

in which

$$\lambda_1' = b_2, \qquad \lambda_n' = b_{2n-1} b_{2n} = \frac{B_{n-2} B_n}{B_{n-1}^2}, \qquad d_n' = b_{2n} + b_{2n+1}.$$

We conclude therefore that the secondary conditions in (30) and (31) can be dispensed with if in the one case only the even convergents of (4) are considered and in the other only the odd convergents.

16. *On the radius of convergence of the series.*—It is clear that the series which satisfy (30) and (31) respectively permit of similar treatment. We shall confine ourselves here to those which satisfy the former condition.

The necessary and sufficient conditions for convergence can then be obtained as follows. It is known that when $A_n > 0$ for all values of n, any minor formed symmetrically from A_n by deleting corresponding rows and columns must also be positive.* Hence $c_{2n} > 0$ and

$$\begin{vmatrix} c_{2n} & c_{2n+2} \\ c_{2n+2} & c_{2n+4} \end{vmatrix} > 0, \qquad \text{i. e.,} \qquad \frac{c_{2n+4}}{c_{2n+2}} > \frac{c_{2n+2}}{c_{2n}}.$$

If, therefore, c_{2n+2}/c_{2n} increases without limit, the series is divergent. On the other hand, *if it increases to a limit* L^2, it can be shown that *the radius of convergence of* (29) *is equal to* $1/L$. This is true obviously after the odd powers are omitted from the series. But since

$$\begin{vmatrix} c_{2n} & c_{2n+1} \\ c_{2n+1} & c_{2n+2} \end{vmatrix} > 0,$$

$|c_{2n+1}|$ must be less than one or both of the coefficients c_{2n} and c_{2n+2}. The odd powers have then a circle of convergence as great as the even powers, and hence the radius for the entire series must be equal to $1/L$.

Since the first $2i$ terms of the series can be obtained by the expansion of N_{2n}/D_{2n} for sufficiently large values of n, we have from (19)

$$\frac{c_{2i}}{c_{2i-2}} = \frac{\sum_{s=1}^{n} M_s z_s^{2i}}{\sum_{s=1}^{n} M_s z_s^{2i-2}} < \bar{z}^2,$$

*SYLVESTER, Philosophical Magazine, ser. 4. vol. 4 (1852), pp. 140–141.

in which $\bar{z}$ denotes the greatest absolute value of any root of $D_{2n}(z)$. It follows that L is equal to or less than the limiting value of $\bar{z}$. This proves that the radius of convergence is at least as great as the reciprocal of the limit of the absolute value of the numerically largest root of $D_{2n}(z)$ or, in other words, as great as the limiting distance from the x-origin to the nearest root of $D_{2n}(x = 1/z)$.

17. *The region of convergence for the continued fraction.*—The question of the convergence of the continued fraction, unlike that of the series, appears to be a very complicated one. The form of the region of convergence, however, can be almost completely determined. First, it should be shown that the continued fraction converges uniformly over some closed finite region.* The region can then be extended by means of a notable theorem discovered by STIELTJES. † Let, namely, $f_1(z)$, $f_2(z)$, $\cdots$ denote an infinite sequence of functions which are holomorphic within a closed area S, then if $\sum f_n(z)$ is uniformly convergent in some region S_2 lying within S and if, furthermore, $f_1(z) + f_2(z) + \cdots + f_n(z)$ has an upper limit independent of n in any region S_1 which includes S_2 but is contained in the interior of S, the series will also converge uniformly in S_1, and its limit will be holomorphic in the interior of S. For application we will put $f_1(z) + f_2(z) + \cdots + f_n(z)$ equal to the nth convergent (or to the nth term of any particular sequence which we may choose to consider). From equations (19) to (21) it is evident that $|N_n(z)/D_n(z)|$ will have an upper limit independent of n in any regions lying wholly at a finite distance from the real axis. The region of convergence can therefore be extended so as to cover the interior of a half plane. If, furthermore, there is any segment of the real axis which is free from roots of D_n (or roots of the denominator of the nth term of the sequence chosen), the region can be extended over this segment so as to include also the interior of the other half plane.

18. *The common cases of convergence.*—The continued fractions of most frequent occurrence are those in which $|b_n|$ has an upper limit k. These I have previously considered, ‡ and have shown that the continued fraction (4) converges within at least a circle of radius $1/4k$, having its center in the origin. The convergence is uniform within any interior circle, and the principle of STIELTJES stated in § 17 extends the region of convergence over the entire interior of the positive and negative half planes.

* This is, for example, true of our continued fraction if

$$\frac{1}{D_n D_{n+1}} - \frac{b_{n+2}x}{D_{n+1}D_{n+2}} + \frac{b_{n+2}b_{n+3}x^2}{D_{n+2}D_{n+3}} \cdots$$

is absolutely convergent at any one point not on the real axis.

† Loc. cit., § 30. For an extension of this theorem by OSGOOD see the Annals of Mathematics, ser. 2, vol. 3, p. 133.

‡ Loc. cit., p. 477.

A second interesting case is that in which the absolute values of the negative (positive) coefficients of (4) but not the positive (negative) coefficients have an upper limit. The continued fraction can then be reduced to a dependence upon that of STIELTJES, but for this purpose the even and odd convergents must be treated separately. If $b_{2n}b_{2n+1} > 0$, take the even coefficients which generate the continued fraction (32), and suppose that the absolute values of the negative b_n have an upper limit less than C. The coefficients d_{n+1} can be made positive by the substitution of $z' + 2C$ for z. Let the new values be denoted by d'_{n+1}. Then the values of the b_n to correspond will be found from the equations:

$$\lambda_n = b'_{2n}b'_{2n+1}, \qquad b'_2 = b_2 + 2C, \qquad d'_{n+1} = d_{n+1} + 2C = b'_{2n+1} + b'_{2n+2}.$$

Since the value of λ_n remains unaltered in the transformation while $d'_n \equiv d_n$, it follows by easy induction from the last equations that

$$b'_{2n} > |b_{2n}|, \qquad 0 < b'_{2n+1} < |b_{2n+1}|.$$

The even convergents of (18) are therefore also the even convergents of another continued fraction of like form,

$$\frac{1}{z'} + \frac{b'_2}{1} + \frac{b'_3}{z'} + \frac{b'_4}{1} + \cdots, \tag{34}$$

in which $b'_n > 0$.

It is obvious that the convergence or divergence of any sequence of convergents is not affected by our transformation, for the latter is merely a change of origin. Now (34) is only a slightly altered form of the continued fraction of STIELTJES, the alternate convergents of which were shown by him to converge over the entire plane of z' with the exception of the whole or a part of the negative half of the real axis. Our sequence of even convergents was therefore convergent over the plane of z with the exception of the portion of the real axis to the left of the point $z = 2C$.

It should not, however, be hastily assumed that the same is true of the odd convergents. Although the continued fractions (32) and (33) had their origin in a single continued fraction (4), they will not correspond after the transformation to the same continued fraction (34), and thus the two sets of convergents lose their initial property of making together a single continued fraction of the form (4).

This fact can be inferred most easily from the equation:

$$D_{2n-2}(z)N_{2n-1}(z) - zD_{2n-1}(z)N_{2n-2}(z) = \lambda_0\lambda_1\cdots\lambda_{n-1}.$$

This relation uniquely determines N_{2n-1}, D_{2n-1} when the continued fraction (32) satisfied by the even convergents is known. When, now, the substitution

$z = z' + 2C$ is made in (32), a new pair of polynomials must be chosen after the transformation which satisfy the equation

$$D_{2n-2}N'_{2n-1} - z'\,D'_{2n-1}N_{2n-2} = \lambda_0\lambda_1\cdots\lambda_{n-1}.$$

The new pair is evidently connected with the old by the equations

$$N'_{2n-1} = N_{2n-1} + t_n N_{2n-2}, \qquad z'\,D'_{2n-1} = zD_{2n-1} + t_n D_{2n-2},$$

in which t_n is a constant which must be so chosen that the absolute term shall disappear in $zD_{2n-1} + t_n D_{2n-2}$, when expressed in terms of z'.

The new rational fractions $N'_{2n-1}/z'D'_{2n-1}$ are the odd convergents of the continued fraction of STIELTJES given in (34) and form in consequence a convergent sequence. The equation

$$\frac{N_{2n-1}}{zD_{2n-1}} = \frac{N'_{2n-1} - t_n N_{2n-2}}{z'D'_{2n-1} - t_n D_{2n-2}}$$

now makes it plain that when (34) is convergent, N_{2n-1}/zD_{2n-1} is, in general, also convergent, but *an exceptional case is evidently possible.** No definite conclusion, then, is reached concerning the sequence of the odd convergents of (4) by means of the transformation.

When $B_n > 0$, the rôle of the odd and even convergents is interchanged.

19. *The less common cases.*—The cases in which neither the positive nor the negative b_n have numerically an upper limit offer greater difficulty. Two interesting cases of this description can be derived by transformation from the continued fraction of STIELTJES. For this purpose place $z = -w^2$ in (1). By a simple reduction it can be brought into the form

$$(35) \qquad -\frac{1}{w}\left[\frac{1}{w} - \frac{b_2}{w} - \frac{b_3}{w} - \cdots\right] \qquad \left(b_n = \frac{1}{a_n a_{n-1}}\right).$$

We can then regard it as a special case of (32) in which $d_n = 0$. Hence its convergents are the *even* convergents of another continued fraction (34) in which $b'_{2n}b'_{2n+1} = \lambda_n = b_{n+1}$ and $b'_{2n+2} = -b'_{2n+1} + C$, if C is properly chosen.

The two cases distinguished by STIELTJES must now be considered separately. When $\sum a_n$ is convergent, $\lim (1/a_n a_{n+1}) = \lim b'_{2n}b'_{2n+1} = \infty$. Since also $b'_{2n+2} = -b'_{2n+1} + C$, there can be no limit numerically to the positive and nega-

* This is in agreement with the fact that in discussing the "problem of moments" STIELTJES finds that the continued fraction for

$$\int_0^\infty \frac{f(z+\lambda)}{z+u+\lambda}\,d\lambda$$

may be divergent while that for

$$\int_0^\infty \frac{f(z)\,dz}{z+u}$$

is convergent; loc. cit., § 78.

tive coefficients in (34), and it is therefore such a continued fraction as we wished to consider. Now STIELTJES has shown that when $\sum a_n$ is convergent in his continued fraction, its even and odd convergents have distinct limits, and these limits expressed in terms of z are meromorphic functions of *genre* 0. In correspondence with this the even convergence of (34) split up into two sequences, each sequence having a limit which is meromorphic in the plane of $w = z' + C$ and is of *genre* 1 or 0. In regard to the odd convergents of (34), we have seen in § 6 that the roots of D_{2n+1} with an increase of n move toward the origin of the plane of $w = 1/x$. Since, moreover, these roots are separated by those of D_{2n+2} which for alternate values of n approach as their limiting positions the poles of two meromorphic functions, the distance between two consecutive roots of D_{2n+1} in any segment of the axis must have a finite lower limit, however much n be increased. The conditions of § 13 therefore are fulfilled. Hence the odd convergents furnish only a single limiting function, and it is easy to infer that this function is meromorphic and of *genre* 1 or 0.

When $\sum a_n$ is divergent, the continued fraction of STIELTJES is convergent, and hence also the even convergents of (34), if derived in the manner just indicated. The latter continued fraction, however, is not necessarily one in which the absolute values of the coefficients have no upper limit. To secure a case of this sort we shall so restrict the a_n that the analytic function which is the limit of (1) can not be continued analytically across any portion of the negative half of the real axis. STIELTJES has shown that this is possible.† By the transformation $z = -w^2$ the half axis is replaced by the entire axis, and the latter is therefore a natural boundary for the limits of the even convergents of (34). Now by § 18 this is impossible if $|b_n|$ is limited. Since, furthermore, $b'_{2n+1} = -b'_{2n+2} + C$, neither the positive elements nor the absolute values of the negative elements will have an upper limit. We conclude then that the continued fraction (34) falls into the category to be considered.

Each point of the axis of $w \equiv z' + C$ must be a point of condensation of the roots of the $2n$th convergent of (34), $(n = 1, 2, \cdots)$. For were this not so, the limit of this convergent by § 17 could be continued analytically across the axis. It follows then by § 12 that (34) is convergent. The corresponding series $P(1/z')$ is divergent; for if it were convergent we could pass by means of the transformation

$$-(z' + C)^2 = -w^2 = z$$

† Loc. cit., § 59. This is a consequence of his equation

$$\lim_{\epsilon=0} [\Phi(-x + \epsilon i) - \Phi(-x - \epsilon i)] = \pi i [\varphi(\bar{x}) + \varphi(\check{x})],$$

which gives the difference of the limiting values of the continued fraction when a point $-x$ of the negative axis is approached from opposite sides. If $\varphi(x)$, the integrand of (3), is non-analytic, so is the integral $\Phi(x)$.

to the series (2) which corresponds to the continued fraction of STIELTJES, and this series would be convergent. But the series (2) was shown by STIELTJES to be divergent, when his continued fraction represents a function incapable of analytic continuation across any portion of the real axis. We arrive thus at the interesting result that *a convergent continued fraction* (34) *and the corresponding divergent series may define in the positive and negative half planes two distinct analytic functions having the real axis as a common natural boundary.*

It should be noted that the two analytic functions take conjugate imaginary values in two points situated symmetrically with respect to the axis. *Our continued fraction thus offers a method of continuing the analytic function across the natural boundary.* The character of this continuation seems to be in essential accord with the ideas of BOREL and FABRY for such a continuation.

IV. *On the Table of Approximants for the Series.*

20. *The table of* PADÉ.—In the previous sections it has been found convenient to consider separately the even and the odd convergents. The essential reason for this will appear if a table of approximants for a series

$$c_0 + c_1 x + c_2 x^2 + \cdots \quad (36)$$

is formed after the manner of PADÉ.†

Among all the rational functions in which the degrees of the numerators and denominators do not exceed two prescribed integers, p and q respectively, there is one rational function and, taken in its lowest terms only one, whose expansion in a series of ascending powers of x agrees with (36) for as great a number of successive terms as possible. Such a fraction I shall term an *approximant* and denote by U_{pq}/V_{pq}. PADÉ arranges all the approximants in a table of double entry:

	$p=0$	1	2	
$q=0$	$\frac{U_{00}}{V_{00}} = c_0$	$\frac{U_{10}}{V_{10}} = c_0 + c_1 x$	$\frac{U_{02}}{V_{02}} = c_0 + c_1 x + c_2 x^2$	$\cdots$
$q=1$	$\frac{U_{01}}{V_{01}}$	$\frac{U_{11}}{V_{11}}$	$\frac{U_{21}}{V_{21}}$	$\cdots$
$q=2$	$\frac{U_{02}}{V_{02}}$	$\frac{U_{12}}{V_{12}}$	$\frac{U_{22}}{V_{22}}$	$\cdots$
	. . . . .			

* The reader will note that the sign of c_{2i-1} has been changed. This has been done to facilitate comparison with the work of FROBENIUS hereafter.

† Thesis: *Sur la représentation approchée d'une fonction par des fractions rationnelles*, Annales de l'Ecole Normale, sér. III, vol. 9 (1892), supplément.

Generally the series for U_{pq}/V_{pq} agrees with (36) for its first $p+q+1$ terms. This is always the case when the fraction, taken in its lowest terms, appears but once in the table. When no rational fraction occurs more than once in the table, the table is said by PADÉ to be normal. The necessary and sufficient condition for the existence of such a table is that none of the determinants

$$c_{\alpha\beta} = \begin{vmatrix} c_{\alpha-\beta+1} & c_{\alpha-\beta+2} & \cdots & c_{\alpha} \\ c_{\alpha-\beta+2} & c_{\alpha-\beta+3} & \cdots & c_{\alpha+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{\alpha} & c_{\alpha+1} & \cdots & c_{\alpha+\beta-1} \end{vmatrix} \qquad (\alpha \geqq 0, \quad c_i = 0 \text{ if } i < 0)$$

shall vanish.*

Obviously all the approximants which lie upon a line perpendicular to the principal diagonal of the table belong to the same value of $p+q+1$. If, then, the table is normal, they each give the same number of successive terms in (36). Hence they may be said to approximate to the series in equal degree or to be "*equally advanced*" in the table.

When the table is not normal, all approximants which are equal to one another fill up a square whose sides are parallel to those of the table. If any such square contains n^2 terms, it may be called an irregularity of the $(n-1)$th order.

Each approximant in the table is surrounded by eight others, unless it lies upon the border when it is enclosed by five. These are called the *contiguous* approximants.

21. *The regular continued fractions.*—Consider now any normal table, and let any succession of approximants be taken, beginning with a term on the border and passing always from one approximant to another which is contiguous to it but more advanced in the table. PADÉ shows that such a sequence of approximants are the successive convergents of a continued fraction, all of whose elements are polynomials in x. This holds also for a non-normal table, provided that in every succession of equal approximants which occur in the sequence all but the first are erased.

The continued fraction is called *regular* when its partial numerators are all of the same degree and likewise its denominators, certain specified and simple irregularities being permitted in the first one or two partial fractions. In a normal table a regular fraction is obtained † when the approximants lie

(1) upon a horizontal or vertical line;

(2) upon a straight line parallel to or coincident with the principal diagonal;

* PADÉ, loc. cit., § 31.

† Loc. cit., § 53.

(3) upon a step-like line, proceeding alternately one term horizontally to the right and one vertically downwards.

Only the second and third types of continued fractions concern us here. If these begin with the pth approximant of the horizontal border, they have the form

$$(2') \quad c_0 + c_1 x + \cdots + c_{p-1} x^{p-1} + \left(\frac{c_p x^p}{1 + l_1 x} + \frac{k_2 x^2}{1 + l_2 x} + \frac{k_3 x^2}{1 + l_3 x} + \cdots \right),$$

$$(3') \quad c_0 + c_1 x + \cdots + c_{p-1} x^{p-1} + \left(\frac{c_p x^p}{1} + \frac{b_2 x}{1} + \frac{b_3 x}{1} + \cdots \right);$$

if with the $p+1$th approximant of the vertical border,

$$(2'') \quad \frac{U_{0,p}}{V_{0p}} + \frac{k_2 x^{p+1}}{1 + l_2 x} + \frac{k_3 x^2}{1 + l_3 x} + \cdots,$$

$$(3'') \quad \frac{U_{0p}}{V_{0p}} + \frac{b_2 x^{p+1}}{1} + \frac{b_3 x}{1} + \cdots.$$

To determine the coefficients of these continued fractions it will be convenient to take $c_0 = 1$ and then to put the constant terms of U_{pq} and V_{pq} equal to 1. The numerators and denominators of the convergents thereby are made identical with certain pairs of polynomials, U_{pq} and V_{pq}, so selected that each pair is followed by another more advanced but contiguous to it. Hence the coefficients in any partial fraction will be the coefficients of some relation connecting three consecutive polynomials V_{pq}. The relations of this character have been determined by FROBENIUS* and run as follows:

$$(37) \quad V_{\alpha+1,\beta} = V_{\alpha\beta} - \frac{c_{\alpha,\beta-1} c_{\alpha+1,\beta+1}}{c_{\alpha,\beta} c_{\alpha+1,\beta}} x V_{\alpha,\beta-1},$$

$$(38) \quad V_{\alpha+1,\beta+1} = V_{\alpha+1,\beta} - \frac{c_{\alpha\beta} c_{\alpha+2,\beta+1}}{c_{\alpha+1,\beta} c_{\alpha+1,\beta+1}} x V_{\alpha\beta},$$

$$(39) \quad V_{\alpha+1,\beta+1} = \left[1 + \left(\frac{c_{\alpha+1,\beta+2} c_{\alpha,\beta-1} - c_{\alpha+2,\beta+1} c_{\alpha-1,\beta}}{c_{\alpha\beta} c_{\alpha+1,\beta+1}} \right) x \right] V_{\alpha\beta} - \frac{c_{\alpha-1,\beta-1} c_{\alpha+1,\beta+1}}{c_{\alpha\beta}^2} x^2 V_{\alpha-1,\beta-1}.$$

*Crelle's Journal, vol. 90 (1881), p. 1, equations (4), (5), and (8) of § 2. These equations have been appropriately modified so as to make the coefficient of $V_{\alpha\beta}$ equal to 1. When taken with the analogous equations for the $U_{\alpha\beta}$, they give the continued fractions of PADÉ for a normal table. PADÉ, however, was the first to note the advantage and significance of the arrangement of the approximants into a table of double entry, and the importance of the various questions which attach themselves to such a table.

Equations (37) and (38) serve to determine the values of the b_n and equation (39) the values of the k_n and l_n.

A comparison of our fundamental continued fraction (4) with (3″) shows that it is of the third type. Since also it is regular from the very outset, it starts from the corner of the table, and the line of approximants which form its convergents proceeds downward in the following manner:

$$\frac{U_{00}}{V_{00}}$$

$$\frac{U_{01}}{V_{01}} \quad \frac{U_{11}}{V_{11}} \qquad (D_{2n+1}(x) = V_{nn}, \quad D_{2n} = V_{n-1,\,n})$$

$$\frac{U_{12}}{V_{12}} \quad \frac{U_{22}}{V_{22}}$$

$$\frac{U_{23}}{V_{23}} \cdots.$$

The odd convergents fill the principal diagonal of the table and the even convergents a parallel file, and hence they form separately two continued fractions of the second type. In the case considered by STIELTJES the two files were shown to converge to distinct limits when $\sum a_n$ is convergent.* Thus his work, when put into relation with PADÉ's table, brings to light the important fact that the table may give rise to continued fractions which represent different functions.† The extension of his work found in the present paper shows also that it is possible for one file to converge and not the other, or even for different sets of convergents chosen from the same file to converge to different limits.

22. *Character of the table when* $\Delta_n > 0$.—It has been already noted that our continued fraction (4) defines uniquely a series and therefore a table of approximants. We proceed now to examine the character of the table when $\Delta_n > 0$. For this purpose it will be necessary to fix the signs of the determinants $c_{\alpha\beta}$ for which $\alpha - \beta$ is an odd integer.

If $\alpha - \beta$ is odd and positive or is equal to -1, the determinant $c_{\alpha\beta}$ is a minor of Δ_n formed symmetrically from its rows and columns, and hence by the theorem of SYLVESTER previously cited must be positive.

Suppose next that $\alpha - \beta = -3$. From a familiar theorem for the product of a determinant by one of its minors we obtain

* In the case of the exponential function PADÉ has shown that the different continued fractions lead to a common limit, Annales de l'Ecole Normale, ser. III, vol. 16 (1899).

† The bearing of the work of STIELTJES has been pointed out previously by PADÉ; Comptes Rendus, vol. 132 (1901), p. 912. No further examination of the table is made as in the present paper.

$$c_{i,\,i+3}\,A_{i+1} \equiv \begin{vmatrix} 0 & 0 & c_0 & \cdots & c_i \\ 0 & c_0 & c_1 & \cdots & c_{i+1} \\ c_0 & c_1 & c_2 & \cdots & c_{i+2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_i & c_{i+1} & c_{i+2} & \cdots & c_{2i+2} \end{vmatrix} \cdot \begin{vmatrix} c_0 & c_1 & \cdots & c_i \\ c_1 & c_2 & \cdots & c_{i+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_i & c_{i+1} & \cdots & c_{2i} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & c_0 & \cdots & c_{i-1} \\ 0 & c_0 & c_1 & \cdots & c_i \\ c_0 & c_1 & c_2 & \cdots & c_{i+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{i-1} & c_i & c_{i+1} & \cdots & c_{2i} \end{vmatrix} \cdot \begin{vmatrix} c_0 & c_1 & \cdots & c_{i+1} \\ c_1 & c_2 & \cdots & c_{i+2} \\ \cdot & \cdot & \cdots & \cdot \\ c_{i+1} & c_{i+2} & \cdots & c_{2i+2} \end{vmatrix} - \begin{vmatrix} 0 & c_0 & \cdots & c_i \\ c_0 & c_1 & \cdots & c_{i+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_i & c_{i+1} & \cdots & c_{2i+1} \end{vmatrix}^2$$

$$\equiv c_{i-1,\,i+2}\,A_{i+2} - (\,)^2.$$

If therefore $c_{i-1,\,i+2}$ is negative, $c_{i,\,i+3}\,A_{i+1}$ and hence $c_{i,\,i+3}$ must also be negative. But $c_{0,\,3} = (-c_0)^3$. Since this is negative, $c_{1,\,4}$ must be negative, then $c_{2,\,5}$, and so on. In other words, $c_{i,\,i+3}$ is negative for all values of i.

It thus appears that the sign of $c_{i,\,i+2n+1}$ is independent of i when $n = 1$ and is $(-1)^n$. We will now show that if this is true for all values of n from 1 to m inclusive, it must be true for $n = m + 1$ and hence universally true. For proof we shall use the same theorem as before for the product of a determinant and its minor. This gives in general

$$c_{i,\,i+2m+3}\,c_{i,\,i+2m+1} = c_{i-1,\,(i-1)+2m+3}\,c_{i+1,\,(i+1)+2m+1} - (c_{i,\,i+2m+2})^2.$$

If, therefore, $c_{i-1,\,(i-1)+2m+3}$ and $c_{i+1,\,(i+1)+2m+1}$ have opposite signs, $c_{i,\,i+2m+3}$ and $c_{i,\,i+2m+1}$ have also opposite signs. But the second and fourth of these quantities have the sign $(-1)^m$ by hypothesis. Hence if $c_{i-1,\,(i-1)+2m+3}$ for any value of i has the sign $(-1)^{m+1}$, the value of i can be increased by successive units without destroying the property. But for the smallest value of i we have

$$c_{0,\,2m+3} = \begin{vmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 0 & 0 & \cdots & c_0 & c_1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ c_0 & c_1 & \cdots & c_{2m+1} & c_{2m+2} \end{vmatrix} = (-1)^{m+1} c_0^{2m+3}.$$

This completes the proof that $c_{i,\,i+2n+1}$ has the sign $(-1)^n$ for all values of i and n.

When $B_n > 0$, it can be shown in similar manner that $c_{\alpha,\,\beta}$ is positive if $\alpha - \beta$ is an even positive integer or 0, and has the sign $(-1)^n$ if $\alpha - \beta = -2n$.

When therefore the two conditions $A_n > 0$, $B_n > 0$ hold simultaneously, $c_{\alpha\beta}$ can not vanish for any value of $\alpha - \beta$, positive or negative. We thus arrive at the conclusion:

THEOREM VIII.—*The table of approximants for the series of* STIELTJES *is a normal one.*

When the condition B_n is dropped, the normal character of the table is not always conserved. For one or more even values of $\alpha - \beta$ the determinant $c_{\alpha\beta}$ may vanish, and in consequence some of the continued fractions furnished by the table may become illusory. It will be noticed, however, that in (39) the difference of the subscripts, $\alpha - \beta$, is the same not only for the three polynomials V_{pq} which there appear but also for the $c_{\alpha\beta}$ occurring in the denominators of the coefficients. The equation therefore has a significance for any odd value of $\alpha - \beta$, and hence the alternate diagonal files of the table for which $\alpha - \beta$ is odd furnish continued fractions of the second type. In these continued fractions it is apparent that no two consecutive convergents can be identical. Therefore the irregularities which appear in the table, being in the form of identical approximants, can contain only 2^2 elements. In other words:

THEOREM IX.—*If* $A_n > 0$ (*or* $B_n > 0$), *all irregularities in the table must be of the first order.*

23. *The zeros and infinities of the approximants.*—We proceed now to the study of the continued fractions furnished by the alternate diagonal files of the table. These have the form (2′) or (2″), the values of the coefficients being obtained from (39). For example, the $(2p-1)$th diagonal file to the right of the principal diagonal gives

$$(40) \qquad c_0 + c_1 x + c_2 x^2 + \cdots + c_{2p-1} x^{2p-1} + \cfrac{c_{2p} x^{2p}}{1 + l_2 x + \cfrac{k_2 x^2}{1 + l_3 x + \cdots}}$$

in which

$$k_n = -\frac{c_{\alpha-1,\beta-1} c_{\alpha+1,\beta+1}}{c_{\alpha\beta}^2} \qquad (\alpha = n + 2p - 1, \quad \beta = n).$$

Now we have seen that when $i - j$ is odd, c_{ij} has a sign depending only upon the subscript difference. Therefore k_n is negative, the relation (39) is analogous to (8), and the continued fraction (40), after the omission of the irregularity at the outset, is precisely such a continued fraction as was satisfied by the even convergents of (4). Our conclusions regarding these convergents were based upon the relation (8) and not upon the form of (4). Accordingly we may express nth convergents of (40) as

$$\frac{N_n'}{D_n'} = c_0 + \cdots + c_{2p-1} x^{2p-1} + \frac{c_{2p} x^{2p} N_{2n}}{D_{2n}}$$

in which N_{2n}/D_{2n} denotes a fraction having the same properties as the $2n$th convergent of (4).

The roots of $D_n' \equiv D_{2n}$, as we have previously seen, are real. To ascertain the number of real roots of N_n' it will be necessary to know first the number of such roots of

$$c_0 + c_1 x + \cdots + c_{2p-1} x^{2p-1} = 0 .$$

Elsewhere* I have shown that only one root of this equation is real, and this was done by proving that

$$\frac{c_0 + c_1 x + \cdots + c_{2p-1} x^{2p-1}}{x^{2p-1}} \tag{41}$$

is an increasing function of $z = 1/x$ and therefore a decreasing function of x. Now xN_{2n}/D_{2n} is either always an increasing or always a decreasing function of x because the roots of numerator and denominator separate each other. But the fraction is zero at the origin and positive for small positive values of x. It must then be an increasing function. Hence as x traverses the real axis, starting from the origin and returning finally to it from the opposite side, xN_{2n}/D_{2n} will become equal to the decreasing function (41) once and but once between two consecutive roots of D_{2n} and once between the origin and each adjoining root. Consequently the number of real roots of N_n' is exactly $n+1$, and $2p-2$ roots will be imaginary. The real roots alternate with the roots of D_n' on each side of the origin, and the root closest to the origin on either side is a root of N_n'.

We turn now to the approximants on the $(2p-1)$th diagonal file below the principal diagonal. It is easy to see that the reciprocals of the approximants are themselves approximants to the expansion (formally) of

$$\frac{1}{(c_0 + c_1 x + c_2 x^2 + \cdots)}$$

in ascending powers of x. The latter can be obtained to any desired number of terms by expanding $V_{n+1,n}/U_{n+1,n}$ into series for sufficiently large values of n. But this fraction, in accordance with the preceding paragraph, has only real zeros and infinities which are situated relatively to one another in the same manner as those of N_{2n}/D_{2n}. Hence its decomposition into partial fractions gives such a sum as

$$\sum \frac{M_i}{z + z_i} \qquad (z = 1/x),$$

M_i being positive as in (19). Now this gives rise to a series satisfying the condition $A_n > 0$.† *The reciprocal of the series* (36) *has therefore precisely the*

*Annals of Mathematics, July, 1903.

†STIELTJES, loc. cit., § 8.

same character as the series itself. Hence $2p-2$ roots of the denominator of any approximant in the $(2p-1)$th diagonal file below the center of the table will be imaginary, while the roots of the numerator alternate on each side of the origin with the real roots of the denominator. This gives

THEOREM X.—*If* $A_n > 0$ *and* $q-p=2n-1$, *the number of imaginary roots of* V_{pq} *will be* $2n-2$ *and its real roots also will alternate with the roots of* U_{pq} *on each side of the origin. If* $p-q=2n-1$, *the polynomials* U_{pq}, V_{pq} *are to be interchanged.*

WESLEYAN UNIVERSITY, MIDDLETOWN, CONN.

ON THE VARIATION OF THE ARBITRARY AND GIVEN CONSTANTS IN DYNAMICAL EQUATIONS*

BY

ERNEST WILLIAM BROWN

Introduction.

The method of the variation of arbitrary constants is a well-known device for obtaining the solution of a set of differential equations when we know that of another set which differs from the former in a suitable manner. Especially is it useful in most of the problems of celestial mechanics where methods of approximation have to be adopted. The first approximation once obtained, it gives the second approximation with ease and certainty, and the same idea can be followed out for the higher approximations if further accuracy is desired.

In the class of problems under consideration here, the equations of motion can be put into the form

$$\frac{d^2x_i}{dt^2} = \frac{\partial F}{\partial x_i} \qquad (i=1, 2, \cdots n),$$

or into the form, known as canonical,

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial X_i}, \qquad \frac{dX_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad (i=1, 2, \cdots n),$$

where F is a function either of the x_i or of the x_i, X_i, respectively, and may or may not contain the time explicitly. Suppose that the problem has been solved so that the x_i, X_i are expressed in terms of $2n$ arbitrary constants and the time, and that it be required to find the solution of another problem in which $F + R$ replaces F. The new problem can be stated by supposing the arbitrary constants in the first problem to be variable and finding the differential equations satisfied by them.

The function F will usually contain, besides x_i, t, a number of given constants which will naturally be present in the solution, and it may happen that some of these constants are really variables which, in order to get a first approximation, we have treated as constants. It is, of course, not possible in general

* Presented to the Society under a different title April 25, 1903. Received for publication April 20, 1903.

to solve the equations as though they were constant and then substitute their variable values in the solution. If we follow the method mentioned we may ask what variable values are to be given to the arbitraries in order that the variable values of the given constants may be substituted in the solution. Thus instead of adding R to F the new problem is that obtained by replacing some of the constants present in F by functions of the time; it is chiefly this problem that I propose to consider in the following pages.

In order to do so it is necessary to consider in some detail the relations satisfied by the derivatives of the dependent variables, not only with respect to the arbitraries but also with respect to the given constants. In the course of this investigation, generalizations of certain theorems which I gave in 1897 * (which included generalizations of ADAMS' theorems connecting the mean motions of the perigee and node with the constant term in the expression for the moon's parallax) are obtained and some new relations between the various constants are added (§§ 1–4). In § 5 the equations for the variable values of the arbitraries corresponding to variable values for the given constants are obtained. It there appears that NEWCOMB's theorem (see art. 24) is a particular case of a general theorem which applies to all equations of dynamics satisfying the conditions laid down. In § 6 applications of the results obtained are made to the lunar theory and in particular to the effects which variable values for the given constants produce on the motions of the arguments.

§ 1. THE EQUATIONS OF VARIATIONS.

1. Let (x, y, z) be the coördinates, referred to fixed axes, of a particle moving under a force-function F. The equations of motion are

$$\frac{d^2x}{dt^2} = \frac{\partial F}{\partial x}, \qquad \frac{d^2y}{dt^2} = \frac{\partial F}{\partial y}, \qquad \frac{d^2z}{dt^2} = \frac{\partial F}{\partial z}. \tag{1}$$

The solution of these equations involves a certain number of constants: of these, six are arbitrary and the others are known and present in F.

Denote by f, g any two of these constants and differentiate (1) with respect to f, g successively. We find

$$\frac{d^2}{dt^2}\frac{dx}{df} = \frac{\partial^2 F}{\partial x^2}\frac{dx}{df} + \frac{\partial^2 F}{\partial x \partial y}\frac{dy}{df} + \frac{\partial^2 F}{\partial x \partial z}\frac{dz}{df} + \frac{d'}{df}\frac{\partial F}{\partial x},$$

$$\frac{d^2}{dt^2}\frac{dx}{dg} = \frac{\partial^2 F}{\partial x^2}\frac{dx}{dg} + \frac{\partial^2 F}{\partial x \partial y}\frac{dy}{dg} + \frac{\partial^2 F}{\partial x \partial z}\frac{dz}{dg} + \frac{d'}{dg}\frac{\partial F}{\partial x},$$

with similar equations for the other coördinates; the operators d'/df, d'/dg

* *On certain properties of the mean motions*, etc., Proceedings of the London Mathematical Society, vol. 28 (1896), pp. 143-155.

denote derivatives with respect to f, g only in so far as they occur explicitly in F.

Multiply these equations by dx/dg, $-dx/df$ respectively, and add the corresponding results for the other two coördinates. Denoting by S a sum of similar expressions with respect to the three coördinates, we obtain an equation which may be written,

$$(2) \qquad \frac{d}{dt}(f, g) = S\left(\frac{dx}{dg}\frac{d'}{df'}\frac{\partial F}{\partial x} - \frac{dx}{df}\frac{d'}{dg}\frac{\partial F}{\partial x}\right) = \frac{d}{dg}\frac{d'F}{df'} - \frac{d}{df}\frac{d'F}{dg},$$

where

$$(3) \qquad (f, g) = S\left(\frac{dx}{dg}\frac{d}{dt}\frac{dx}{df} - \frac{dx}{df}\frac{d}{dt}\frac{dx}{dg}\right) = -(g, f).$$

If h be another of the constants we have, by direct differentiation,

$$(4) \qquad \frac{d}{df}(g, h) + \frac{d}{dg}(h, f) + \frac{d}{dh}(f, g) = 0.$$

2. The constants will now be separated into two classes. Suppose that the solution is such that F and each coördinate is expressible by means of sums of cosines *or* sines of sums of multiples of the angles

$$b_i t + \beta_i = w_i,$$

where b_i and the coefficients of the periodic terms depend only on one set of constants α_i, the other set being β_i, and the α_i, β_i being all independent of one another; these will be respectively called the α-set and the β-set. It is supposed that F is given in terms of x, y, z and the known constants, involving the latter in the form just given. It is also supposed that the expressions for F and for its derivatives contain no term independent of x, y, z. Also the constant of length is to be such a function of the arbitrary constants of the solution that x, y, z, the kinetic energy T, and F contain no term independent of the arbitraries. In general, we shall avoid choosing the constant of length as an arbitrary so that certain functions which are only defined by their derivatives with respect to the arbitraries need no added constant independent of them.

3. Let α, α' be two constants of the α-set, and β, β' two of the β-set corresponding to angles w, w'. Denote by $\partial/\partial\alpha$ derivatives with respect to α only in so far as α is contained in the coefficients, so that d/df refers to derivatives when α_i, β_i are considered independent and $\partial/\partial f$ to derivatives when α_i, w_i are considered independent: evidently $\partial/\partial w_i = \partial/\partial\beta_i = d/dw_i = d/d\beta_i$.

Denote by $\dot{x}, \dot{y}, \dot{z}$ derivatives with respect to t. Since $\dot{x} = \Sigma_i b_i \, dx/d\beta_i$, we have

$$\frac{dx}{d\alpha} = \frac{\partial x}{\partial \alpha} + t\Sigma_i \frac{\partial x}{\partial w_i}\frac{db_i}{d\alpha},$$

$$\frac{d}{dt}\left(\frac{dx}{d\alpha}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial \alpha}\right) + \Sigma_i \frac{\partial x}{\partial w_i}\frac{db_i}{d\alpha} + t\Sigma_i \frac{\partial \dot{x}}{\partial w_i}\frac{db_i}{d\alpha}$$

$$= \frac{\partial \dot{x}}{\partial \alpha} + t\Sigma_i \frac{\partial \dot{x}}{\partial w_i}\frac{db_i}{d\alpha},$$

in which it is to be noted that $\partial/\partial\alpha$, d/dt are not commutative although $\partial/\partial w_i$, d/dt are so.

Substituting these results in (3) and putting

$$[f, g] = S\left(\frac{\partial \dot{x}\,\partial x}{\partial f\,\partial g} - \frac{\partial x\,\partial \dot{x}}{\partial f\,\partial g}\right) = -[g, f],$$

we obtain

$$(\alpha, \alpha') = [\alpha, \alpha'] + t\Sigma_i \left\{ \frac{db_i}{d\alpha}[w_i, \alpha'] - \frac{db_i}{d\alpha'}[w_i, \alpha] \right\} + t^2 \Sigma_i \Sigma_j \frac{db_i}{d\alpha}\frac{db_j}{d\alpha}[w_i, w_j],$$

(5)
$$(\alpha, \beta) = [\alpha, w] + t\Sigma_i \frac{db_i}{d\alpha}[w_i, w],$$

$$(\beta, \beta') = [w, w'].$$

Also

$$\frac{\partial}{\partial f}[g, h] + \frac{\partial}{\partial g}[h, f] + \frac{\partial}{\partial h}[f, g] = 0. \tag{6}$$

The quantities (f, g) must be divided into several classes, each of which requires separate consideration. The constants f, g may both belong to the α-set, or to the β-set, or one to the α-set and the other to the β-set. Again, they may both be arbitrary, or both known, or one arbitrary and one known. I shall only consider two known constants — one belonging to each set — so that there will be eight different kinds of brackets to consider, three of them being independent of the known constants. The extension to any number of known constants is made without difficulty.

§ 2. Application to the Arbitrary Constants.

4. When the constants considered in § 1 are the arbitraries of the solution, the right hand member of (2) is zero since F does not contain them explicitly; all the (f, g) are therefore constant. But $[\alpha, \alpha']$, $[\beta, \beta']$ consist only of

sums of sines owing to the forms assumed for the coördinates. Hence by § 1, art. 3,

$$(7)\quad (\alpha, \alpha') = [\alpha, \alpha'] = (\beta, \beta') = [w, w'] = 0, \quad (\alpha, \beta_i) = [\alpha, w_i],$$

$$(8)\qquad \Sigma_i \frac{db_i}{d\alpha} [\alpha', w_i] = \Sigma_i \frac{db_i}{d\alpha'} [\alpha, w_i].$$

Also from (6),

$$\frac{\partial}{\partial \alpha} [\alpha', w] - \frac{\partial}{\partial \alpha'} [\alpha, w] = \frac{\partial}{\partial w} [\alpha', \alpha] = 0;$$

hence

$$(9)\quad [\alpha_i, w_j] = (\alpha_i, \beta_j) = \frac{dc_j}{d\alpha_i} = \frac{\partial^2}{\partial \alpha_i \partial w_j} (c_1 w_1 + c_2 w_2 + c_3 w_3) \quad (i, j = 1, 2, 3),$$

where c_1, c_2, c_3 are functions of the arbitrary and known constants of the α-set only; they are thus completely defined in accordance with the remark at the end of art. 2.

Equation (8) then gives

$$\Sigma_i \frac{db_i}{d\alpha} \frac{dc_i}{d\alpha'} = \Sigma_i \frac{db_i}{d\alpha'} \frac{dc_i}{d\alpha},$$

or

$$\frac{d}{d\alpha} \left(\Sigma_i b_i \frac{dc_i}{d\alpha'} \right) = \frac{d}{d\alpha'} \left(\Sigma_i b_i \frac{dc_i}{d\alpha} \right),$$

showing that

$$(10)\qquad \Sigma_i b_i dc_i = -dB,$$

a perfect differential. This gives a complete definition of B if we assume that it contains no term independent of the arbitrary constants.

5. *Canonical systems.*—The constants α_i can in general be chosen arbitrarily. Take $\alpha_i = c_i$. Then

$$[c_i, w_i] = (c_i, \beta_i) = 1, \qquad [c_i, w_j] = (c_i, \beta_j) = 0 \qquad (j \neq i).$$

The choice is possible if the Jacobian of the c_i with respect to the α_i is not zero: in the applications to be made it is easy to see that this is the case. Thus the c_i are assumed to be independent. We then have from art. 4,

$$b_i = -\frac{dB}{dc_i},$$

where B is expressed in terms of the c_i and the known constants.

All these results are perfectly well known. The sets c_i, β_i and c_i, w_i are canonical systems, for the (f, g), $[f, g]$ are nothing else than LAGRANGE's brackets and the above values reduce LAGRANGE's equations for the variation

of arbitrary constants, when a disturbing function R is added to F, to the canonical forms

$$\frac{dc_i}{dt} = \frac{dR}{d\beta_i}, \qquad \frac{d\beta_i}{dt} = -\frac{dR}{dc_i},$$

and

$$\frac{dc_i}{dt} = \frac{dR'}{dw_i}, \qquad \frac{dw_i}{dt} = -\frac{dR'}{dc_i},$$

where $R' = R + B$ and R is supposed to be expressed in terms of the w_i, c_i, t and known constants.

6. Another canonical set is obtained by putting

$$B_i = \frac{dc_1}{d\alpha_i}\beta_1 + \frac{dc_2}{d\alpha_i}\beta_2 + \frac{dc_3}{d\alpha_i}\beta_3.$$

and taking α_i, B_i as arbitraries; this is evident from equation (9). The same equation shows that if we put

$$W_i = \frac{dc_1}{d\alpha_i}w_1 + \frac{dc_2}{d\alpha_i}w_2 + \frac{dc_3}{d\alpha_i}w_3,$$

the α_i, W_i form a canonical set.

The sets c_i, β_i; α_i, B_i; α_i, W_i have the disadvantage that when we form the equations of variations the time appears as a factor of the periodic terms. This fact prevents them from being useful for most purposes of the lunar theory to which application will chiefly be made. It is to be noticed nevertheless that when β_i or w_i are taken as one set of arbitraries, the c_i is the conjugate set which makes the whole canonical; if α_i be taken as one set then the B_i or W_i has the same property with reference to the α_i. In general, if one set of constants be given the second set which renders the whole canonical is determined uniquely; the apparent duplicity in the above is due to the fact that W_i, w_i are not arbitrary *constants* but contain the time.

If it be possible to choose the b_i as one set of arbitraries, the conjugate sets are

$$\Sigma_j \beta_j \frac{dc_j}{db_i} = \Sigma_j \beta_j \frac{dc_i}{db_j}, \qquad \Sigma_j w_j \frac{dc_j}{db_i} = \Sigma_j w_j \frac{dc_i}{db_j} \quad (j=1, 2, 3);$$

and from (10),

$$c_i = \frac{d}{db_i}(B + \Sigma_j b_j c_j),$$

so that

$$(b_i, \beta_j) = [b_i, w_j] = \frac{d^2}{db_i\, db_j}(B + b_1c_1 + b_2c_2 + b_3c_3).$$

§ 3. Application to the Known Constants.

7. Suppose that F contains a known constant α_0 of the α-set and a known angle $w_4 = b_4 t + \beta_4$ in which (the more general case) b_4 is a function of α_0 and β_4 belongs to the β set. Then applying (6) successively to the various combinations of α_0, w_4 with the arbitraries and using (7), (9) we obtain

$$\frac{\partial}{\partial \alpha'}[\alpha, w_4] - \frac{\partial}{\partial \alpha}[\alpha', w_4] = \frac{\partial}{\partial w_4}[\alpha, \alpha'] = 0,$$

$$\frac{\partial}{\partial \alpha'}[\alpha, \alpha_0] - \frac{\partial}{\partial \alpha}[\alpha', \alpha_0] = \frac{\partial}{\partial \alpha_0}[\alpha, \alpha'] = 0,$$

$$\frac{\partial}{\partial \alpha}[w, w_4] - \frac{\partial}{\partial w}[\alpha, w_4] = \frac{\partial}{\partial w_4}[w, \alpha] = 0,$$

$$\frac{\partial}{\partial \alpha}[\alpha_0, w] = \frac{\partial}{\partial \alpha_0}[\alpha, w] - \frac{\partial}{\partial w}[\alpha, \alpha_0],$$

$$\frac{\partial}{\partial \alpha}[\alpha_0, w_4] = \frac{\partial}{\partial \alpha_0}[\alpha, w_4] - \frac{\partial}{\partial w_4}[\alpha, \alpha_0].$$

The first three of these give

$$(11) \qquad [\alpha_i, w_4] = \frac{\partial C_4}{\partial \alpha_i}, \quad [\alpha_i, \alpha_0] = \frac{\partial C_0}{\partial \alpha_i}, \quad [w_i, w_4] = \frac{\partial C_4}{\partial w_i} \quad (i = 1, 2, 3),$$

and the last two then give

$$(12) \qquad [\alpha_0, w_i] = \frac{dc_i}{d\alpha_0} - \frac{\partial C_0}{\partial w_i}, \qquad [\alpha_0, w_4] = \frac{\partial C_4}{\partial \alpha_0} - \frac{\partial C_0}{\partial w_4} \quad (i = 1, 2, 3),$$

in which C_4, C_0 are, owing to the forms of the square brackets as deduced from the expressions for the coördinates, trigonometric series with the arguments w_i, w_4 and with coefficients depending on α_i, α_0; C_4 consists of cosines only, C_0 of sines only. Moreover, C_4 is only defined by its derivates with respect to α_0, α_i, w_i and may therefore contain terms independent of them, while C_0 is only defined by its derivatives with respect to α_i, w_i, w_4 and might contain terms independent of them. The suppositions of § 1 prevent the presence of such terms, so that C_0, C_4 are fully defined.

If the solution of the original equations has been obtained algebraically we have then sufficient material to find C_4, C_0. It may happen, however, that numerical values have been given to the constants in forming the solution, and for many purposes it is useful to possess equations for C_4, C_0 which will enable us to find their values in this case. It will be shown that they satisfy ordinary differential equations of the first order.

These equations may be obtained directly from the results of arts. 3, 4, 7 without using the original equations of motion. But as they involve $\partial' F/\partial \alpha_0$, $d' F/\partial w_4$, it is much easier to obtain them directly from the results of arts. 4, 7 and the equations of motion. This latter method, which also furnishes the vis viva equation, will be followed.

In this section, unless explicitly stated otherwise, the suffixes i, j receive only the values 1, 2, 3, that is, the letters to which they are attached are *arbitraries* of the solution.

8. *The Vis Viva equation.* Since

$$[\alpha_i, w_j] = \frac{dc_j}{d\alpha_j}, \qquad [\alpha_i, w_4] = \frac{dC_4}{d\alpha_i},$$

$$\frac{dQ}{dt} = \Sigma_i b_i \frac{dQ}{dw_i} + b_4 \frac{dQ}{dw_4},$$

where Q contains the time only through its presence in the angles w_i, w_4, we have

$$\Sigma_j b_j \frac{dc_j}{d\alpha_i} + b_4 \frac{dC_4}{d\alpha_i} = S\left(\frac{\partial \dot{x}}{\partial \alpha_i}\dot{x} - \ddot{x}\frac{\partial x}{\partial \alpha_i}\right) = \frac{\partial}{\partial \alpha_i}(T - F).$$

But $\Sigma_j b_j dc_j = -dB$. Therefore, since there are no terms independent of the arbitrary constants and b_4 is a known constant,

$$T - F - b_4 C_4 = -B. \tag{13}$$

9. *The equation for C_4.* Differentiating (13) with respect to the time,

$$b_4 \frac{dC_4}{dt} = \frac{d}{dt}(T - F) = S(x\dot{x} - \ddot{x}\dot{x}) - b_4 \frac{d' F}{dw_4},$$

or

$$\frac{dC_4}{dt} = -\frac{\dot{d' F}}{dw_4}. \tag{14}$$

The variable C_4 is therefore similar to the L of M. POINCARÉ in his paper *Sur les équations de la lune.** We might have defined C_4 by means of this equation and have then reversed the processes of this article and obtained the first and third of equations (11). As C_4 can contain a constant term, the definition is not complete, but will be made so by the equation obtained in art. 9.

* Bulletin Astronomique, vol. 17 (1900), p. 167.

10. *The equation for* C_0.—Multiply the first of equations (12) by b_i, the second by b_4 and form the sum. We obtain

$$\Sigma_i b_i \frac{dc_i}{d\alpha_0} + b_4 \frac{\partial C_4}{\partial \alpha_0} - \frac{dC_0}{dt} = S\left(\frac{\partial \dot{x}}{\partial \alpha_0} x - \ddot{x} \frac{\partial x}{\partial \alpha_0}\right)$$
$$= \frac{\partial T}{\partial \alpha_0} - \frac{\partial F}{\partial \alpha_0} + \frac{\partial' F}{\partial \alpha_0}$$
$$= \frac{\partial}{\partial \alpha_0}(b_4 C_4) - \frac{dB}{d\alpha_0} + \frac{\partial' F}{\partial \alpha_0},$$

by (13). In this equation B is supposed to be expressed in terms of α_i, α_0. Let $dB_c/d\alpha_0$ denote the derivative of B with respect to α_0 when B is expressed in terms of c_1, c_2, c_3, α_0, so that

$$\frac{dB}{d\alpha_0} = \frac{dB_c}{d\alpha_0} + \Sigma_i \frac{dB_c}{dc_i}\frac{dc_i}{d\alpha_0} = \frac{dB_c}{d\alpha_0} - \Sigma_i b_i \frac{dc_i}{d\alpha_0}.$$

Substituting in the former equation and rearranging, we obtain

$$\frac{dC_0}{dt} + C_4 \frac{db_4}{d\alpha_0} - \frac{dB_c}{d\alpha_0} = -\frac{\partial' F}{\partial \alpha_0}, \tag{15}$$

the equation required.

Since C_0 contains no constant term, (15) gives it completely when C_4 has been obtained. We already know C_4 from (14) except its constant term. But since C_0 contains no term proportional to the time, the constant term in C_4 is given by

$$\frac{db_4}{d\alpha_0}(C_4)^0 - \frac{dB_c}{d\alpha_0} = -\left(\frac{\partial' F}{\partial \alpha_0}\right)^0, \tag{15}^0$$

where $(\cdot\cdot)^0$ denotes the constant term in the expansion of the enclosed function in trigonometric series.

If b_4 does not depend on α_0, it is easily seen that the equation has its second term absent and C_0 can be found independently of C_4.

But now the constant part of C_4 cannot be determined by $(15)^0$. The difficulty is turned by following the investigation of this article with b_4 as the constant instead of α_0. Then as b_4 occurs explicitly in F only through its presence in m_4 the equation (15) reduces to

$$\frac{dC_0}{dt} + C_4 - \frac{dB_c}{db_4} = 0,$$

giving

$$(C_4)^0 = \frac{dB_c}{db_4}.$$

The C_0 of this equation is, of course, different from that of (15).

There may be other known constants α_p present in F. If b_4 depends on α_p then there will be an equation for C_p corresponding to (15). If b_4 is independent of α_p, the equation satisfied by the corresponding C_p is

$$\frac{dC_p}{dt} - \frac{dB_c}{d\alpha_p} = -\frac{\partial' F}{\partial \alpha_p}, \tag{16}$$

where C_p consists of sines only. It will be convenient in what follows to suppose that any other constants are all of the latter type and therefore may either be treated as numerical constants in which case the corresponding equation (16) will be unnecessary, or as algebraic constants in which case it may be necessary to use the equation. Both cases may arise in the method of the variation of known constants which is developed below.

11. Suppose that F is homogeneous of order q with respect to x, y, z, and a certain constant A present in F, there being also present an absolute constant (e. g., the "gravitation" constant) which restores the proper physical dimensions of the equations of motion.

Then we have, symbolically,

$$F,\ B,\ T = q\ (\text{length}).$$

The equations of motion show that

$$\text{time} = (1 - \tfrac{1}{2}q)\ (\text{length}),$$

and since $[c_i, w_i] = 1$, $b = -1$ (time),

$$c_i = (1 + \tfrac{1}{2}q)\ (\text{length}) = \frac{q+2}{q-2}(b_i), \qquad F,\ B,\ T = \frac{2q}{q+2}(c_i).$$

The choice of α_0 has been quite arbitrary. When it is a function of b_4 take it so that it is of the same dimensions as the c_i and further such that

$$\frac{d\alpha_0}{db_4} = \frac{q+2}{q-2}\frac{\alpha_0}{b_4}.$$

As A is merely defined as a length, choose it so that

$$\frac{dA}{A} = \frac{2}{q+2}\frac{d\alpha_0}{\alpha_0}.$$

Thus A, α_0 are defined in terms of b_4, but the manner in which they occur in F is still arbitrary except with respect to their dimensions.

Since B will be homogeneous with respect to the c_i, α_0 we have

$$\alpha_0 \frac{dB_c}{d\alpha_0} = \frac{2q}{q+2}B - \Sigma_i c_i \frac{dB}{dc_i} = \frac{2q}{q+2}B + \Sigma_i b_i c_i.$$

12. Multiply (15) by $q+2$ and apply the results for dA, $d\alpha_0$ in art. 11. We obtain

$$-(q+2)\alpha_0\frac{dC_0}{dt}-(q-2)b_4C_4+(q+2)\alpha_0\frac{dB_c}{d\alpha_0}=2A\frac{\partial' F}{\partial A},$$

where F is supposed to be expressed in terms of A instead of α_0.

Now F is homogeneous of order q with respect to x, y, z, A. Hence, multiplying equations (1) by $2x$, $2y$, $2z$, and adding their sum to the last equation, we obtain

$$\frac{d^2}{dt^2}r^2-4T-(q+2)\alpha_0\frac{dC_0}{dt}-(q-2)b_4C_4+(q+2)\alpha_0\frac{dB_c}{d\alpha_0}=2qF,$$

which, by (15) and the last result in art. 11, becomes

$$\frac{d^2}{dt^2}r^2-(q+2)\alpha_0\frac{dC_0}{dt}-(q-2)(T-F+B)+2qB$$
$$+(q+2)\Sigma b_ic_i=2qF+4T,$$

or, reducing,

$$T+F-B-b_1c_1-b_2c_2-b_3c_3=\frac{d}{dt}\left(\frac{1}{2+q}\frac{d}{dt}r^2-\alpha_0C_0\right). \tag{17}$$

Denote by zero index the constant terms in the expansion of these functions. Then, since the right-hand member contains no constant term,

$$T^0+F^0=B+b_1c_1+b_2c_2+b_3c_3, \tag{18}$$

a result independent of q.

13. The equation (18) gives a useful formula for finding the c_i when the solution has been obtained with any set of arbitraries and even when numerical values have been given to the α-set. For

$$c_i=\frac{\partial}{\partial b_i}(B+\Sigma b_ic_i)=\frac{\partial}{\partial b_i}(T^0+F^0)$$
$$=S\left[\dot{x}\frac{\partial\dot{x}}{\partial b_i}+\ddot{x}\frac{\partial x}{\partial b_i}\right]^0=S\left[\dot{x}\frac{dx}{dw_i}+\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial b_i}\right)\right]^0.$$

or

$$c_i=\left[\dot{x}\frac{\partial x}{\partial w_i}+\dot{y}\frac{\partial y}{\partial w_i}+\dot{z}\frac{\partial z}{\partial w_i}\right]^0 \qquad (i=1,2,3). \tag{19}$$

Define a new constant c_4 by a similar equation for $i=4$. We then have

$$b_1c_1+b_2c_2+b_3c_3+b_4c_4=2T^0,$$

or by (18),

$$T^0-F^0-b_4c_4=-B.$$

Comparing with the vis viva equation, we obtain

$$c_4=(C_4)^0. \tag{20}$$

§ 4. Canonical Equations of Motion.

14. The equations (14), (15) may be written

$$(21) \qquad \frac{dC_4}{dt} = -\frac{\partial' F}{\partial w_4}, \quad \frac{d}{dt}\left(C_0 - t\frac{\partial B_c}{\partial \alpha_0}\right) = -\frac{\partial' F}{\partial \alpha_0} - C_4\frac{db_4}{d\alpha_0}.$$

These with the equations of motion (1) suggest the following transformation. Put

$$2T = \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \qquad W_0 = -C_0 - b_0 t + \beta_0, \qquad \phi = T - F - b_4 C_4,$$

where β_0 is a constant. If we consider x, y, z, C_4, α_0 and $\dot{x}$, $\dot{y}$, $\dot{z}$, w_4, W_0 as independent variables and ϕ expressed as a function of them, the equations (1) and (21) may be written in the canonical form

$$(22) \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \phi}{\partial \dot{x}}, & \dfrac{d\dot{x}}{dt} = -\dfrac{\partial \phi}{\partial x}; \ \cdots, \ \cdots; \ \cdots, \ \cdots; \\ \dfrac{dC_4}{dt} = \dfrac{\partial \phi}{\partial w_4}, & \dfrac{dw_4}{dt} = -\dfrac{\partial \phi}{\partial C_4}; \end{cases}$$

$$(23) \qquad \frac{d\alpha_0}{dt} = \frac{\partial \phi}{\partial W_0}, \qquad \frac{dW_0}{dt} = -\frac{\partial \phi}{\partial \alpha_0};$$

for ϕ does not contain W_0 explicitly, α_0 is a constant, and $dw_4/dt = b_4$. The same equations hold if b_4 is not a function of α_0.

We may either use the first four or all five pairs of equations and the solutions of such canonical systems are well known. The last pair gives $\alpha_0 =$ constant and defines W_0. It will be more convenient, however, to treat α_0 as a given constant and use only the first four equations in the following work.

When (22) are solved canonically in terms of the constants c_j and the angles w_j $(j = 1, 2, 3, 4)$ we find

$$\phi = \text{constant} = -B', \qquad b_j = -\frac{dB'}{dc_j}.$$

The previous articles show that $B' = B + b_4 c_4$ where B is independent of c_4 [the assumption that the constant parts of ϕ, etc., contain no parts independent of c_1, c_2, c_3 no longer holds]. Hence, as before, $b_i = -\partial B_4/\partial c_i$. Also the expressions for the coördinates are independent of c_4.

15. All the preceding investigations might have been developed by starting with these canonical equations and solving them by means of Jacobi's method, obtaining the results in the inverse order. I have preferred to follow the more natural if somewhat longer development as it seems to make the deduction of some of the results rather more clear in view of the applications which follow.

If there be angles other than w_4 and constants other than α_0 present in F, similar results may be obtained. In fact, the extension of this method to any number of variables, angles and constants will be sufficiently obvious. To each angle and constant corresponds respectively a pair of canonical equations; to ϕ must be added one term for each angle present. When this has been done, ϕ, considered as a function of all the variables, is independent of t explicitly and $\phi = \text{const.}$ is an integral of the equations.

§ 5. Variation of Constants.

16. *Variation of arbitrary constants.*—Let a new function ψ be added to ϕ, then, using the canonical set $c_1, c_2, c_3, c_4, w_1, w_2, w_3, w_4$, the equations satisfied by them are

$$\frac{dc_j}{dt} = -\frac{\partial}{\partial w_j}(\psi - B') = -\frac{\partial \psi}{\partial w_j}, \qquad \frac{dw_j}{dt} = \frac{\partial}{\partial c_j}(\psi - B') \quad (j = 1, 2, 3, 4),$$

where ψ and B' are expressed in terms of c_j, w_j, of the given constants and the time, B' being a function of the c_j and the constants present in ϕ.

The method appears to fail if ψ contains none of the arbitraries c_j, w_j. But the equations of motion depend on the *derivatives* of ψ with respect to the variables; we can therefore assume that ψ is not independent of the arbitraries.

If ψ is independent of the w_j, the c_i are constant and the equations for w_j are easily solved. If ψ contains a small factor α and a term with argument $bt + \beta$, where b is small and β independent of the β_i, the corresponding terms in w_i will contain divisors $b, b^2/\alpha, b^3/\alpha^2, \cdots$; when β is not independent of β_j the first divisor will in general be b^2.

17. *Variation of the given constants.* Let ψ be due to a small addition $\delta\alpha_0$ to α_0, one of the constants present in F where $\delta\alpha_0$ may be a constant or a function of the time. Then using the canonical equations (22) and assuming that $\delta\alpha_0$ produces variations δc_j, δw_j and that squares of the variations may be neglected,

$$\psi = \Sigma_j \left(\frac{\partial \phi}{\partial c_j}\delta c_j + \frac{\partial \phi}{\partial w_j}\partial w_j + \frac{\partial \phi}{\partial \alpha_0}\delta\alpha_0 \right) = -\Sigma_j \frac{dB'}{dc_j}\delta c_j - \frac{dB'}{d\alpha_0}\delta\alpha_0,$$

since B' is independent of the w_j.

The equations of variations are

$$\frac{d}{dt}\delta c_j = -\frac{\partial \psi}{\partial w_j} = \Sigma_{j'} \frac{dB'}{dc_{j'}}\frac{\partial}{\partial w_j}\delta c_{j'} + \frac{dB'}{d\alpha_0}\frac{\partial}{\partial w_j}\delta\alpha_0,$$

$$\frac{d}{dt}(w_j + \delta w_j) = \frac{\partial \psi}{\partial c_j} + b_j = -\frac{\partial}{\partial c_j}\left(\Sigma_{j'} \frac{dB'}{dc_{j'}}\delta c_{j'} + \frac{dB'}{d\alpha_0}\delta\alpha_0 \right) + b_j$$

$$(j, j' = 1, 2, 3, 4).$$

As squares of the variations are neglected we can substitute constant values for the coefficients of the variations in the right hand members and therefore

$$\frac{d}{dt}\delta c_j = -\Sigma_{j'} b_{j'} \frac{\partial}{\partial w_j}\delta c_{j'} + \frac{dB'}{d\alpha_0}\frac{\partial}{\partial w_j}\delta\alpha_0,$$

$$\frac{d}{dt}(w_j + \delta w_j) = \frac{\partial}{\partial c_j}\left(\Sigma_{j'} b_{j'}\delta c_{j'} - \frac{dB'}{d\alpha_0}\delta\alpha_0\right) + b_j.$$

18. Suppose that $\delta\alpha_0$ is independent of the w_j, c_j. The last terms of the equations for δc_j then vanish and they are satisfied by

$$\delta c_j = 0.$$

Substituting in the second set we find, since $B' = B + b_4 c_4$,

$$\frac{d}{dt}(w_j + \delta w_j) = \frac{db_j}{d\alpha_0}\delta\alpha_0 + b_j.$$

That is, in order to obtain the new value of w_j, all that is necessary is to substitute $\alpha_0 + \delta\alpha_0$ for α_0 in the expression for b_j expressed as a function of c_1, c_2, c_3, α_0.

Suppose that the original solution had been expressed in terms of any arbitraries $\alpha_1, \alpha_2, \alpha_3$ of the α-set, and of the known constant α_0; the c_i having been obtained in terms of them by the method of art. 13. Then the variations of α_i are found by solving the equations

$$\frac{dc_i}{d\alpha_1}\delta\alpha_1 + \frac{dc_i}{d\alpha_2}\delta\alpha_2 + \frac{dc_i}{d\alpha_3}\delta\alpha_3 = -\frac{dc_i}{d\alpha_0}\delta\alpha_0 \qquad (i=1,2,3),$$

for $\delta\alpha_1, \delta\alpha_2, \delta\alpha_3$. The changes in the coefficients of the periodic terms are obtained by substituting these changes of the constants therein. But the changes in the b_i are given by

$$\delta b_i = \frac{db_i}{d\alpha_1}\delta\alpha_1 + \frac{db_i}{d\alpha_2}\delta\alpha_2 + \frac{db_i}{d\alpha_3}\delta\alpha_3 + \frac{db_i}{d\alpha_0}\delta\alpha_0 \qquad (i=1,2,3),$$

that is to say, we must also substitute the change of α_0 as well as those of $\alpha_1, \alpha_1, \alpha_3$ in the b_i.

The same results hold if b_4 is not a function of α_0.

19. If ψ be due to a variation δw_4 of w_4, we have in a similar manner,

$$\psi = \Sigma_j b_j \delta c_j;$$

and the equations of variations are

$$\frac{d}{dt}\delta c_j = -\Sigma_{j'} b_{j'}\frac{\partial}{\partial w_j}\delta c_{j'}, \qquad \frac{d}{dt}(w_j + \delta w_j) = -\frac{\partial}{\partial c_j}\left(\Sigma_j \frac{dB'}{dc_{j'}}\delta c_{j'}\right) + b_j.$$

Suppose that the variation of w_i is given. The left hand member of the last of the second set, namely,

$$\frac{d}{dt}\delta w_i = \Sigma_{j'} b_{j'} \frac{\partial}{\partial c_i}\delta c_{j'}$$

is known.

The value of δw_i may consist of periodic and non-periodic terms: suppose it to be independent of the angles w_1, w_2, w_3 and of c_1, c_2, c_3. Let us try if $\delta c_1 = 0$, $\delta c_2 = 0$, $\delta c_3 = 0$ and δc_4 independent of w_1, w_2, w_3 can satisfy the equations. The first three of the first pair are satisfied and the others become

$$\frac{d}{dt}\delta c_i = -b_i \frac{\partial}{\partial w_i}\delta c_4 \quad \frac{d}{dt}\delta w_4 = b_4 \frac{\partial}{\partial c_4}\delta c_4, \quad \frac{d}{dt}\delta w_i = b_i \frac{\partial}{\partial c_i}\delta c_4 \quad (i=1, 2, 3).$$

If $\delta w_4 = qt + q'$, where q, q' are given constants, the equations are satisfied by $\delta c_4 = qc_4/b_4$, $\delta w_i = 0$. It does not appear to be possible in general to satisfy the equations with other values for δw_4.

§ 6. Applications to the Lunar Theory.

20. The results contained in the previous sections are more general in their character than is necessary for the applications to most problems in celestial mechanics and in particular for that one which is known as the lunar theory. Without going into details it is sufficient to state here that the few restrictions imposed are satisfied in these applications.

Let x, y, z, r denote the coördinates and distance of the moon relatively to the earth's center; x', y', z', r', those of the sun relatively to the center of mass of the earth and moon; $\mu = E + M$, the sum of the masses of the earth and moon; m' the mass of the sun; let the three bodies be considered to attract in the same way as particles of the same masses situated at their centers of mass respectively. Also suppose that the motion of the sun is known and that all other influences are neglected. Then it is well known that

$$F = \frac{\mu}{r} + m'\{r^2 - 2r(xx' + yy' + zz') + r'^2\}^{-\frac{1}{2}} - m'(xx' + yy' + zz')r'^{-3},$$

where, after the substitution of polar coördinates for x', y', z' and expansion in powers of $1/r'$, the term $1/r'^3$ is dropped and the coefficient of $(r')^{-p}$ is multiplied by

$$\kappa_p = \left(\frac{E}{E+M}\right)^{p-1} - \left(\frac{-M}{E+M}\right)^{p-1}.$$

We can therefore put

$$F = \frac{\mu}{r} + m' \sum_{p=2}^{\infty} \kappa_p R'_p (r')^{-p-1},$$

where R'_p is a homogeneous function of order p with respect to x, y, z, and also

contains trigonometric functions of the angular position of the sun. Further, if n', a' be the mean motion and mean distance of the sun, $m' + \mu = n'^2 a'^3$, or, putting $\lambda = \mu/(m' + \mu)$,

$$m' = n'^2 a'^3 (1 - \lambda).$$

We shall now suppose that the plane of the sun's orbit is fixed and that its motion can always be expressed in terms of n', e' and the angle $n't + \beta_4$; when its motion is elliptic n', β_4, e' are constants. Properly, there are two angles $n't + \epsilon'$, $n't + \epsilon' - \pi'$, but well known results show that we may treat them as though π' were zero and they also indicate where π' has to be inserted in the final results. Thus we can put

$$F = \frac{\mu}{r} + n'^2 (1 - \lambda) \sum_{p=2}^{\infty} \frac{R_p}{a'^{p-2}},$$

where R_p is homogeneous and of order p relatively to x, y, z and contains κ_p, e' and trigonometric functions of $n't + \epsilon'$.

21. We have from the equations of motion,

$$\tfrac{1}{2} \frac{d^2}{dt^2} r^2 - 2T = Sx \frac{\partial F}{\partial x} = -\frac{\mu}{r} + n'^2 (1 - \lambda) \Sigma_p p R_p (a')^{-p+2} \quad (p = 2, 3, \cdots).$$

Eliminating T between this equation and (17) we find, since $q = -1$ here,

$$(24)\quad \tfrac{3}{2} \frac{\mu}{r} - \tfrac{1}{2} n'^2 (1 - \lambda) \Sigma_p (p - 2) R_p (a')^{-p+2} - B - \Sigma_i b_i c_i = \tfrac{3}{4} \frac{d^2}{dt^2} r^2 - \alpha_0 \frac{dC_0}{dt}.$$

The right hand member contains no constant term. Hence

$$(25)\quad B + b_1 c_1 + b_2 c_2 + b_3 c_3 = \tfrac{3}{2} \left(\frac{\mu}{r} \right)^0 - \tfrac{1}{2} n'^2 (1 - \lambda) \Sigma_p (p - 2) (R_p)^0 (a')^{-p+2}.$$

The equations (14), (15) show that dC_0/dt is a quantity whose lowest order is n'^2/α_0. As it is usual in the lunar theory to expand in powers of n' and $1/a'$, it will be more convenient to put $\alpha_0 C_0 = C_0'$ so that dC_0'/dt is a quantity whose lowest order is n'^2 and is expansible like the coördinates. We therefore have

$$(26)\quad \frac{dC_0'}{dt} = \tfrac{3}{4} \frac{d^2}{dt^2} (r^2) - \tfrac{3}{2} \frac{\mu}{r} + \tfrac{1}{2} n'^2 (1 - \lambda) \Sigma_p (p - 2) R_p (a')^{-p+2} + f \quad (p = 3, 4, \cdots),$$

f being a constant so chosen as to make the constant part of the right-hand member zero.

22. The properties of equations (25), (26) which are remarkable and valuable for practical purposes arise principally in those investigations which permit us to neglect the ratio of the parallaxes of the sun and moon. This ratio is

about $\frac{1}{4}\frac{1}{0}\frac{1}{0}$ and for most of the purposes which require the value of C'_0, i. e., in the calculation of the indirect action of a planet (art. 24 below), it is possible to neglect it. But of the R_ρ, the first term R_2 is the only one which is independent of it and the factor $\rho - 2$ causes R_2 to disappear in both the equations. The equation (26) then reduces to

$$\frac{dC'_0}{dt} = \tfrac{3}{4}\frac{d^2}{dt^2}(r^2) - \tfrac{3}{2}\left(\frac{\mu}{r}\right) + \tfrac{3}{2}\left(\frac{\mu}{r}\right)^0. \tag{27}$$

Another observation may be made. If the object in view is such that we only require those terms in C'_0 which are of long period relatively to the month, i. e., those in which the coefficients of t in the arguments are small, the first term of the right-hand member will give rise to coefficients which are very small compared with those arising from the second member; in such cases we can generally put

$$\frac{dC'_0}{dt} = -\tfrac{3}{2}\left(\frac{\mu}{r}\right) + \tfrac{3}{2}\left(\frac{\mu}{r}\right)^0,$$

without committing sensible errors.

23. In nearly all cases of the indirect action of a planet, we can neglect the *square* of the ratio of the parallaxes. But it is well known that the constant parts of the functions we have been dealing with are expansible in powers of the square of this ratio. *Hence neglecting the square of the ratio of the parallaxes,*

$$B + b_1 c_1 + b_2 c_2 + b_3 c_3 = \tfrac{3}{2}\left(\frac{\mu}{r}\right)^0.$$

This is the result which I obtained in an earlier paper * and which I used to prove ADAMS' well known theorems and to obtain the secular accelerations directly from the expression for the parallax of the moon. It is true that the square of the ratio of the parallaxes was neglected, but it was easy to see from the few terms which DELAUNAY gave in a literal form that they could exert no sensible effect on the result. If doubts might still arise, owing to slow convergence, it is not difficult to remove them by means of the numerical results of my lunar theory taken in conjunction with equation (25).

The methods used here differ in form from those which were adopted in the paper just referred to. The essential feature of the earlier method was really that of DELAUNAY, that is to say, the elliptic orbit was used as a first approximation and the variation of arbitrary constants applied to it somewhat in DELAUNAY's manner. Here we are theoretically quite independent of the elliptic orbit, and any set of arbitrary constants which possesses the property of enabling

* See footnote, p. 334 above.

us to expand in trigonometric series may be used. This has the practical advantage of permitting the calculation of c_1, c_2, c_3, by means of art. 13 from the results of any theory in whatever form it may be expressed and whatever be the arbitrary constants used. Provisionally it may be best to except the theory of HANSEN, although it is probable that the expression $\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z$, which is the principal part of equation (19), might be transformed so as to be used in this case also without serious difficulty.

24. The results of arts. 15, 16 include NEWCOMB's theorem concerning the secular accelerations and the indirect action of a planet * as a particular case. In this valuable memoir he considered the action of a planet as a case of the disturbed motion of three bodies, that is to say the problem of the relative motion of the earth, moon and sun is considered to have been solved and the question was then to find out what variable values the 12 arbitrary constants which arise in the solution must have in order that the solution may retain the same form when another disturbing cause acts. He showed that if we first consider the orbit of the sun as elliptic and if the action of a planet causes variations $\delta e'$, $\delta n'$ in e', n', then the new non-periodic parts of n, π_1, θ_1 (the mean motions of the moon, its perigee and node) are found by solving the equations

$$\frac{\partial c_i}{\partial n}\delta n + \frac{\partial c_i}{\partial e}\delta e + \frac{\partial c_i}{\partial \gamma}\delta\gamma + \frac{\partial c_i}{\partial n'}\delta n' + \frac{\partial c_i}{\partial e'}\delta e' = 0 \qquad (i = 1, 2, 3)$$

for δn, δe, $\delta\gamma$, regarding $\delta n'$, $\delta e'$ as known, and substituting the variable values of n, e, γ, n', e' in the expressions of n, π_1, θ_1 in terms of these five quantities. He remarks, "This curious theorem may embody some principle applicable to the disturbed motion of three bodies, which has not yet been fully mastered."

It appears from the investigation of this paper that the theorem is independent of the particular form in which the solar coördinates enter into the lunar disturbing function and further, that it is true for any of the constants which may be present in this disturbing function, provided always that the given variations of the known constants are independent of the lunar elements. Finally, it is applicable to any dynamical problem in which the coördinates satisfy the specified conditions and to any constants which may enter into the force function in the manner specified at the outset.

HAVERFORD COLLEGE,
April 20, 1903.

* *Action of the Planets on the Moon*, Washington Astronomical Papers, vol. 5 (1894), p. 191.

THE PRIMITIVE GROUPS OF CLASS $2p$ WHICH CONTAIN A SUBSTITUTION OF ORDER p AND DEGREE $2p$*

BY

W. A. MANNING

If a group contains substitutions displacing just N letters, and none besides the identity displacing fewer than N letters, it is of class N.† All the substitutions of class N are regular. If N is a composite number, it may be assumed that the group of class N contains a substitution A composed of l cycles of prime order p.‡

JORDAN determined all the primitive groups of class p.§ He also published without proof a list of the primitive groups of class less than 14.‖ Among these are some of the primitive groups of class $2p$, namely, those of the classes 4, 6, and 10. Proofs of his results for class 4 have since been published.¶ There are six of these groups: the metacyclic G^5_{20}, the semimetacyclic G^5_{10}, the G^6_{120} simply isomorphic to the symmetric group of degree 5, the icosaedral G^6_{60}, the simple G^7_{168}, and the G^8_{1344}. Since these results are well established and offer exceptions to the general theory, it will be assumed in what follows that p is an odd prime.

THEOREM I. *If an intransitive or imprimitive group of degree $lp+1$ and class lp contains a substitution of order p, then must $l \equiv p+2$.*

First let the group be intransitive. It must be a simple isomorphism between its $e+1$ systems of intransitivity. One transitive constituent is of degree

* Presented to the Society (San Francisco) December 20, 1902. Received for publication February 27, 1903.

† C. JORDAN, Comptes Rendus, vol. 72 (1871), p. 384.

‡ C. JORDAN, Liouville's Journal, ser. 2, vol. 16 (1871), p. 390.

§ C. JORDAN, Liouville's Journal, ser. 2, vol. 17 (1872), p. 363.

In addition to the cyclic group of degree p there exists a primitive group of degree $p+1$ and order $p(p+1)$, and one of degree $p+2$ and order $p(p+1)(p+2)$, if, and only if, $p+1$ is a power of 2.

‖ Comptes Rendus, vol. 73 (1871), p. 385.

¶ NETTO, *Theory of Substitutions*, Cole's translation (1892), p. 133.
MILLER, American Mathematical Monthly, vol. 9 (1902), p. 63.

$rp+1$, class rp, and order $(rp+1)r'p$, where r' divides r.* The remaining e constituents are regular and of degree $(rp+1)r'p$, so that

$$lp+1 = er'p(rp+1)+rp+1;$$

whence

$$l = er'(rp+1)+r; \qquad e \geqq 1,\ r \geqq r' \geqq 1.$$

Suppose next that the group is imprimitive. The substitution A of order p leaves one letter u fixed. The system of imprimitivity to which u belongs is transformed into itself by A. If two letters of a cycle of A belong to the same system, all the letters of that cycle belong to the system. Then the number of letters in each system is $rp+1$. The other $l-r$ cycles of A must permute the remaining systems. The number of systems of imprimitivity in the group is therefore $ep+1$. Hence we have $lp+1=(ep+1)(rp+1)$; from which follows $l=e(rp+1)+r$, where $e \geqq 1$, $r \geqq 1$.

In both cases the minimum value of l is $p+2$.

THEOREM II. *Two similar substitutions of prime order and of lowest class in a group, which are not powers the one of the other, and such that the number of letters in a cycle exceeds the number of cycles in each substitution, cannot displace exactly the same letters.*

Let S and T be two similar substitutions of lowest class N which displace the same N letters. Since the number of letters in a cycle exceeds the number of cycles, one of the substitutions contains in one of its cycles two letters a_1 and a_2 which occur in a certain cycle of the other. Then two numbers σ and τ can be so chosen that $S^\sigma T^\tau$ leaves a_1 fixed. But the group containing S and T is of class N and therefore $S^\sigma T^\tau = 1$, which proves the theorem.

THEOREM III. *A primitive group containing a substitution of order p on $2p$ letters includes a transitive subgroup of degree $\leqq 2p+2$ generated by two similar substitutions of order p and degree $2p$.*

The proof of this proposition is contained in a memoir of JORDAN, to which the reader is referred.† An outline of the proof is, however, necessary to make clear the application of the reasoning there employed to this special case.

The primitive group G contains a transitive subgroup H generated by the similar substitutions $A=(a_1a_2\cdots a_p)(b_1b_2\cdots b_p)$, A', A'', $\cdots$. Since H is transitive, among the substitutions A', A'', $\cdots$ there will exist at least one A_1 which permutes letters of the first cycle of A with other letters. Now A_1 cannot replace each of the p letters a by b's and new letters c, for then A_1 must contain

* BURNSIDE, *Theory of Groups* (1897), p. 141.
FROBENIUS, Berliner Sitzungsberichte (1902), p. 455.

†Crelle's Journal, vol. 79 (1874), pp. 249–253. Cf. JORDAN, *Memoire sur les groupes primitifs*, Bulletin de la Société Mathématique de France, vol. 1 (1873), p. 175.

all the a's and must not have more than $(p-1)/2$ of them in each cycle, which is impossible since there are but two cycles. It is next shown that the substitution A_1 can be chosen in such a way that it replaces an a by a b. It may happen that among the substitutions A', A'', $\cdots$ there exist several which replace an a by a b. Choose for A_1 the one that displaces the fewest new letters c. Finally JORDAN proves that A_1 cannot contain more than one c in each of its cycles. The group generated by A and A_1 is transitive and of degree $\equiv 2p+2$.

We notice that this subgroup is positive since it is generated by two positive substitutions. It does not contain a substitution of order 2 on $2p$ letters. If the condition that the primitive group G under consideration is of class $2p$ is introduced, theorem II limits the degree of this subgroup K to $2p+1$ and $2p+2$. Then there are two cases to be considered. We shall take them up separately.

CASE I. The group G contains a transitive subgroup K of degree $2p+1$ and class $2p$. From theorem I it follows that K is primitive. Since it is positive, the subgroups of degree $2p$ are of order p and hence K is of order $(2p+1)p$. There is in K a selfconjugate subgroup F of degree and order $2p+1$.* Moreover $2p+1$ must be a power of a prime,† as q^α, and since

$$2p=(q-1)(q^{\alpha-1}+q^{\alpha-2}+\cdots+q+1),$$

it follows that $q=3$, or $\alpha=1$. When $2p+1$ is a prime, K is the semimetacyclic group. Since no substitution of degree $2p+1$ can be commutative with one of the substitutions of order p, and since no operator of odd order can transform a substitution of order 3 into its inverse, it follows that when $q=3$, F contains p subgroups of order 3, all conjugate in K. Hence F is Abelian and of type $(1,1,\cdots 1)$, and K is a subgroup of the holomorph of this Abelian group F. The order of this holomorph is $3^\alpha(3^\alpha-1)(3^\alpha-3)\cdots(3^\alpha-3^{\alpha-1})$, and it is divisible by $p=(3^\alpha-1)/2$ but not by p^2. Then all the subgroups of order p in the holomorph are conjugate. Therefore all the subgroups of order $(2p+1)p$ are also conjugate. The above reasoning applies to any positive primitive group G of degree $2p+1$ and class $2p$ as no other conditions were imposed upon K. There is then only one positive primitive group of degree $2p+1$ and class $2p$ when $2p+1$ is a power of 3 as well as when $2p+1$ is a prime.

If a positive $G^{(2p+2)}$ (on $2p+2$ letters) of class $2p$ exists, the subgroup leaving one letter fixed is the $G^{(2p+1)}$ just determined, so that $G^{(2p+2)}$ is doubly transitive and of order $(2p+2)(2p+1)p$. Each subgroup of order p is self-

* FROBENIUS, Berliner Sitzungsberichte (1901), p. 1220.

† FROBENIUS, Berliner Sitzungsberichte (1902), p. 458; BURNSIDE, *Theory of Groups* (1897), p. 143.

conjugate in a subgroup of order $2p$, in which the substitutions that are not of order p are of degree $2p+2$. It will now be shown that $G^{(2p+2)}$ cannot exist if the subgroup of order $2p$ is cyclic and $p>3$. Such a group is known to exist if $p=3$, namely, the compound G^{8}_{168}.

A cyclic group of order $2p$ has only one substitution of order 2. No other substitution of order 2 in $G^{(2p+2)}$ can transform the same subgroup of order p into itself, for then the latter would be selfconjugate in a subgroup of higher order than $2p$. To each of the $p+1$ cycles of such a substitution of order 2 there corresponds a subgroup of order p leaving the two letters of that cycle fixed, and with which it is commutative. Therefore $G^{(2p+2)}$ contains just $2p+1$ substitutions of order 2. Now $G^{(2p+2)}$ contains $2(2p+2)p$ substitutions of degree $2p+1$, $(2p+1)(p^2-1)$ of order p and degree $2p$, at least $(2p+1)(p^2-1)$ of order $2p$ and degree $2p+2$, and $2p+1$ of order 2, making with the identity a total of at least $2p(2p+1)(p+1)$ substitutions. But this is the order of the group. No substitutions of prime order other than those enumerated occur. Then $p+1$ must be a power of 2 when $2p+1$ is a prime greater than 3. But $p+1$ is divisible by 3 since neither $2p$ nor $2p+1$ is divisible by 3. If $2p+1=3^{\beta}$, $p+1$ is not divisible by 3 and must again be a power of 2. But both $2^{\alpha}-1=p$ and $2^{\alpha+1}-1=3^{\beta}$ cannot hold. Then $G^{(2p+2)}$ does not exist if the subgroup of order $2p$ is cyclic and $p>3$.

There is one other form which the subgroup of order $2p$ may have. It may be a p, 1 isomorphism between a regular diedral rotation group and the group in the remaining two letters. All the substitutions of a regular diedral rotation group are completely determined by the subgroup of order p. Then $G^{(2p+2)}$, if it exists, is completely determined by the subgroup leaving one letter fixed and is therefore unique. When $2p+1$ is a prime, the modular group on $2p+2$ letters is a positive doubly transitive group of class $2p$. When $2p+1=3^{\beta}$, a group satisfying our conditions is defined by the congruences

$$y \equiv \frac{ax+b}{cx+d}, \mod 3, \qquad ad-bc \equiv 1, \mod 3,$$

in which a, b, c, and d are marks of the $GF[3^{\beta}]$.* The reasoning of this paragraph clearly holds when $p=3$. The group in question is the simple G^{8}_{168} of class 6.

It is known that there is just one positive primitive group of degree 9 and class 6. Its order is 1512 and it contains the compound G^{8}_{168}. This G^{9}_{1512} is not, however, in turn contained in a primitive group of degree 10.†

*MATHIEU, Liouville's Journal, ser. 2, vol. 5 (1860), p. 9; vol. 6 (1861), p. 241.

†COLE, Bulletin of the New York Mathematical Society, vol. 2 (1892), p. 250. MILLER, Quarterly Journal, vol. 31 (1899), p. 228.

‡BURNSIDE, *Theory of Groups* (1897), p. 202. Cf. ib., exercise, p. 204.

Assuming that $p > 3$ we have next to determine whether a positive triply transitive group containing $G^{(2p+2)}$ exist on $2p+3$ letters.‡ In such a group each subgroup of order p would be selfconjugate in a subgroup I of order $6p$. Evidently I has a transitive constituent of degree 3 and order 6, permuting the letters left fixed by the subgroup of order p. This is the symmetric group of degree 3. Then this symmetric group must be found in the quotient group of the largest group on the same letters that transforms a given group of degree $2p$ and order p into itself. But in this quotient group the operators of order 2 generate a subgroup in which there is no operator of order 3. The subgroup I of order $6p$ can therefore not exist. Then when $p > 3$, the chain of primitive groups comes to an end with the degree $2p+2$.

Under this case are still to be considered those groups in which occurs a negative substitution. The primitive group G contains a selfconjugate subgroup of half its order composed of the totality of its positive substitutions, which must be one of the primitive groups just determined.

Of the primitive groups of class 6 neither the compound G^{s}_{168} nor the G^{3}_{1512} is contained in a positive and negative group of the same degree.

The group G is of order $(2p+1)2p$ or $(2p+2)(2p+1)2p$ as its degree is $2p+1$ or $2p+2$. Since the subgroup of degree $2p$ is also of order $2p$, these groups are doubly and triply transitive respectively.

First, let the degree of G be $2p+1$. If this number is a prime, G is metacyclic. If $2p+1=3^{\beta}$, G contains a regular Abelian selfconjugate subgroup F of order 3^{β}, which contains p cyclic subgroups of order 3. The substitution $A=(a_1a_2\cdots a_p)(b_1b_2\cdots b_p)$ permutes these p subgroups cyclically. The subgroup G_1 of G that leaves one letter fixed and contains A is either cyclic or of diedral type. If G_1 is cyclic of order $2p$, it contains just one operator of order 2. A substitution of order 2 which with A generates a transitive cyclic group can be chosen in p ways. This substitution must, however, transform at least one subgroup of order 3 in F into itself. Only one of these p substitutions of order 2 can have this property, and it transforms every substitution of F into its inverse. This gives us the well known doubly transitive group of degree 3^{β} defined by

$$y \equiv ax + b, \quad \text{mod. } 3.$$

where a and b are marks of $GF[3^{\beta}]$, $a \neq 0$.

There remains the possibility that G_1 is a regular diedral rotation group. In this case all the negative substitutions of G are of order 2, and if we select any one of them and multiply it into all the negative substitutions of G in turn, all the positive substitutions of G are obtained. Now any two operators of order 2 generate a group in which the cyclic subgroup generated by their product is selfconjugate. Hence every substitution of order 2 in G transforms every subgroup of order p into itself. This is absurd. Then there is

only the one positive and negative primitive group of degree $2p+1$ and class $2p$, which contains a substitution of order p.

It follows at once that there is not more than one positive and negative triply transitive group of degree $2p+2$; for such a group is determined by its positive subgroup of order $(2p+2)(2p+1)p$ and any negative substitution found in the subgroup leaving one letter fixed. The group exists and is the linear fractional group written on $2p+1$ letters, defined by

$$y \equiv \frac{ax+b}{cx+d},$$

$$ad - bc \not\equiv 0,$$

mod. $2p+1$, when $2p+1$ is a prime; and mod. 3, when $2p+1 = 3^s$, with the additional condition in the latter case that a, b, c, and d run through the marks of the $GF[3^s]$.

CASE II. Suppose next that G is positive and contains a transitive subgroup K of degree $2p+2$ generated by two similar substitutions of order p. We may assume that K is imprimitive and that it is not contained in a primitive group of the same degree. For a primitive group of degree $2p+2$ contains a subgroup of degree $2p+1$ and class $2p$, which we have seen (theorem I) must be primitive, and this gives case I again. In fact K can contain no substitution displacing $2p+1$ letters. Now one of the generators,

$$A = (a_1 a_2 \cdots a_p)(b_1 b_2 \cdots b_p)$$

say, must interchange systems of imprimitivity of K. If two letters of a cycle of A belong to the same system of imprimitivity, all the letters of that cycle belong to the system. Therefore K has $p+1$ systems of two letters each. Every substitution of order p in K determines a system by the two letters which it leaves fixed, and hence the elements of the $p+1$ systems can be chosen in but one way. The subgroup of K that leaves one letter fixed leaves two fixed and is of order p. Then the order of K is $(2p+2)p$. If k is contained in a primitive group of degree n, that group is $n-(2p+2)+1 = n-2p-1$ times transitive,* and must in consequence contain a doubly transitive G of degree $2p+3$. The order of G is clearly $(2p+3)(2p+2)p$. When $p > 3$, a subgroup of order p in G is transformed into itself by a subgroup of order $6p$, which, as we have already seen, is impossible in a primitive group of this degree.

In case $p=3$, G is of degree 9 and order 216. It is not contained in a primitive group of degree 10, but does lead to a $\pm G^{9}_{432}$, which is the holomorph of the regular non-cyclic group of order 9.†

*JORDAN, Liouville's Journal, ser. 2, vol. 16 (1871), p. 383.

MARGGRAFF, *Dissertation, Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades*, Giessen (1892).

†COLE, Bulletin of the New York Mathematical Society, vol. 2 (1892), p. 250.

Summing up our results, we find that there are five positive and three positive and negative primitive groups of class 6 which contain a substitution of order 3 on two cycles. They are the $\pm G^{7}_{42}$, the $\pm G^{8}_{336}$, the compound G^{8}_{168}, the G^{9}_{1512}, the $\pm G^{9}_{432}$ and their positive subgroups. When $p > 3$, primitive groups of class $2p$ containing a substitution of order p and degree $2p$ exist only when $2p + 1$ is a prime or a power of 3. There are just four groups when one or the other of these conditions is satisfied. All four are included in the triply transitive Mathieu group.

The determination of the primitive groups of class $2p$ which contain no substitution of order p on $2p$ letters seems to present greater difficulties than the case here considered.

STANFORD UNIVERSITY,
December, 1902.

COMPLETE SETS OF POSTULATES FOR THE THEORY OF REAL QUANTITIES*

BY

EDWARD V. HUNTINGTON

The following paper presents two complete sets of postulates, or primitive propositions, either of which may be used as a basis for the ordinary algebra of real quantities. †

* Presented to the Society, under a slightly different title, December 29, 1902. Received for publication February 2, 1903. [For an abstract of an unpublished paper on the same subject, presented to the Society by the writer on April 26, 1902, see Bulletin of the American Mathematical Society, vol. 8 (1901–02), p. 371.]

† Cf. D. HILBERT, *Ueber den Zahlbegriff*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), pp. 180–184.—The axioms for real numbers enumerated by HILBERT in this note include many redundancies, and no attempt is made to prove the uniqueness of the system which they define. (Cf. Theorem II below.) The main interest of the paper lies in his new Axiom der Vollständigkeit, which, together with the axiom of Archimedes, replaces the usual axiom of continuity. The other axioms are given also in his *Grundlagen der Geometrie* (1899), § 13.

Complete sets of postulates for particular classes of real quantities (positive integral, all integral, positive real, positive rational) can be found in the following papers:

G. PEANO, *Sul concetto di numero*, Rivista di Matematica, vol. 1 (1891), pp. 87–102, 256–267; Formulaire de Mathématiques, vol. 3 (1901), pp. 39–44.—Here only the positive integers, or the positive integers with zero, are considered. An account of these postulates is given in the Bulletin of the American Mathematical Society, vol. 9 (1902–03), pp. 41–46. They were first published in a short Latin monograph by PEANO, entitled *Arithmetices principia nova methodo exposita*, Turin (1889).

A. PADOA: 1) *Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque*, Bibliothèque du congrès international de philosophie, Paris, 1900, vol. 3 (published in 1901), pp. 309–365; 2) *Numeri interi relativi*, Rivista di Matematica, vol. 7 (1901), pp. 73–84; 3) *Un nouveau système irréductible de postulats pour l'algèbre*, Compte rendu du deuxième congrès international des mathématiciens, Paris, 1900 (published in 1902), pp. 249–256.—The second of these papers is an ideographical translation of the first; the third reproduces the principal results.

E. V. HUNTINGTON: 1) *A complete set of postulates for the theory of absolute continuous magnitude*; 2) *Complete sets of postulates for the theories of positive integral and positive rational numbers*; Transactions, vol. 3 (1902), pp. 264–279, 280–284.—The first of these papers will be cited below under the title: *Magnitudes*. [In the fifth line line of postulate 5, p. 267, the reader is requested to change "one and only one element A" to: *at least one element* A —a typographical correction which does not involve any further alteration in the paper.]

Among the other works which may be consulted in this connection are:

H. B. FINE, *The number system of algebra treated theoretically and historically*, Boston (1891).

O. STOLZ und J. A. GMEINER, *Theoretische Arithmetik*, Leipzig (1901–02). Two sections of this work have now appeared.

G. PEANO, *Aritmetica generale e algebra elementare*, Turin (pp. vii + 144, 1902).

The fundamental concept involved is that of an *assemblage*, or *class*, M, in which *a relation*, $\circledcirc$, and *a rule of combination*, $\oplus$, and (in § 2) *another rule of combination*, $\odot$, are defined. Any object which belongs to the assemblage is called an *element*.

Thus, if a and b are elements of the assemblage, $a \circledcirc b$ indicates that a stands in the given relation to b; while $a \oplus b$ and $a \odot b$ denote the objects uniquely determined by a and b (in their given order), according to the first and second rules of combination respectively.

(It will appear that the symbols

$$\circledcirc, \oplus, \text{ and } \odot$$

obey the same formal laws as the symbols

$$<, +, \text{ and } \cdot$$

of elementary arithmetic, and they may therefore be called by the same names: *less than*, *plus*, and *times*.)

The relation $\circledcirc$ and the rules of combination $\oplus$ and $\odot$ are wholly undetermined, except for the imposition of certain postulates (§ 1 and § 2), which are shown to be *consistent* (Theorems I and I′), and *independent* (Theorems III and III′).

Any assemblage in which $\circledcirc$ *and* $\oplus$ *are so defined as to satisfy the postulates* 1–10 *of* § 1, *or any assemblage in which* $\circledcirc$, $\oplus$, *and* $\odot$ *are so defined as to satisfy the postulates* 1–14 *of* § 2, *shall be called a system of real quantities.* It is shown in Theorems II and II′ that all such systems are *equivalent;* in other words, the system of real quantities is uniquely determined by either of the two sets of postulates.

Each set of postulates is thus a *complete set.**

In the proofs of Theorems II and II′, use is made of certain theorems proved in the writer's paper on the postulates of magnitude.

In the proofs of Theorems I, III, and I′, III′, the existence is assumed of the ordinary system of real numbers — that is, the totality of finite and infinite decimal fractions (or the totality of CANTOR's *Fundamentalreihen* or DEDEKIND's *Schnitte*), built up from the system of positive integers in the usual way, by successive generalizations of the number concept.

The system of *real quantities* and the system of *real numbers* are of course mathematically equivalent concepts; the contrasted use of the terms in this paper is intended to correspond with HILBERT's distinction between the "axiomatic" and the "genetic" *methods of defining* that concept.

* *Magnitudes*, p. 264.

§ 1. The first set of postulates for real quantities.

We consider in the first place an assemblage in which $\circledcirc$ and $\oplus$ are so defined that the following ten postulates are satisfied:

1. *If a and b belong to the assemblage, and $a \neq b$, then either $a \circledcirc b$ or $b \circledcirc a$.*

2. *The relation $\circledcirc$ is transitive; that is, if a, b, c belong to the assemblage, and $a \circledcirc b$ and $b \circledcirc c$, then $a \circledcirc c$.*

3. *The relation $c \circledcirc c$ is false for at least one element c.*

4. *If a and b belong to the assemblage, and $a \circledcirc b$, then there is an element x such that $a \circledcirc x$ and $x \circledcirc b$.*

5. *If S is any infinite sequence of elements (a_k) such that*

$$a_k \circledcirc a_{k+1}, \quad a_k \circledcirc c \qquad (k = 1, 2, 3, \cdots),$$

(where c is some fixed element), then there is an element A having the following two properties:

1°) *$a_k \circledcirc A$ whenever a_k belongs to S;*
2°) *if $A' \circledcirc A$, then there is at least one element of S, say a_r, for which $A' \circledcirc a_r$.*

6. *If a, b and $b \oplus a$ belong to the assemblage, then $a \oplus b = b \oplus a$.*

7. *If a, b, c, $a \oplus b$, $b \oplus c$ and $a \oplus (b \oplus c)$ belong to the assemblage, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.*

8. *For every two elements a and b there is an element x such that $a \oplus x = b$.*

9. *If $x \circledcirc y$, then $a \oplus x \circledcirc a \oplus y$, whenever $a \oplus x$ and $a \oplus y$ belong to the assemblage.*

10. *The assemblage contains at least two distinct elements.**

It will be noticed that postulates 1–5 concern only the relation $\circledcirc$, and postulates 6–8 only the rule of combination $\oplus$.

Consistency of the postulates.

Theorem I.—*The postulates* 1–10 *are consistent with one another;* that is the postulates themselves, and therefore all their consequences, are free from contradiction.

* The necessity of adding this postulate was first pointed out to me by Mr. L. D. Ames. The addition and consequent modifications are made, May, 1903. It is to be noted that in *Magnitudes* the existence of at least one element in the assemblage might better have appeared (p. 267) as an explicit postulate than as a presupposition (p. 266, l. 6).

Proof.—To establish this theorem it is sufficient to prove the existence of some assemblage in which $\odot$ and $\oplus$ are so defined that all the postulates are satisfied.*

One such assemblage (if we admit the axioms of positive integral numbers) is the system of all real numbers, built up from the system of positive integers by the "genetic" method with $\odot$ and $\oplus$ defined as $<$ and $+$; another is the system of *positive* real numbers, with $\odot$ and $\oplus$ defined as $<$ and $\times$.

Still another such system (if we admit the usual axioms of geometry of one dimension) is the system of points on a straight line through a fixed point O, with $\odot$ defined as "on the left of," and $\oplus$ as the $+$ of vector analysis.

Deduction from the postulates.

The following propositions, 15–30, deduced from the fundamental propositions 1–10, will prepare the way for Theorem II.

In the first place, it follows from 6, 7, 8 that our assemblage is an *Abelian group with respect to* $\oplus$.† Hence:

15.‡ If a and b belong to the assemblage, then $a \oplus b$ does also.

16. The element x in 8 is uniquely determined by a and b. $(x = b - a.)$

17. There is one and only one zero-element, 0, such that $a \oplus 0 = a$ for every element a.

Next, from 3, with 6, 8, and 9, we have:

18. The relation $a \odot a$ is false for every element a.

* The question of the consistency (compatibility, Widerspruchslosigkeit) of any set of postulates is emphasized by HILBERT as the most important problem which can be raised concerning them, since the proof of the consistency of the postulates "is at the same time the proof of the mathematical existence" of the concept which they define. The method of proof here used for Theorem I (which is the only method now known) would probably not be satisfactory to HILBERT; for he writes: "While the proof of the compatibility of the geometrical axioms may be made to depend upon the theorem of the compatibility of the arithmetical axioms, the proof of the compatibility of the arithmetical axioms requires on the contrary a direct method"—that is, an absolute method depending only on the consideration of the axioms themselves, and not involving the introduction of an auxiliary system whose "mathematical existence" might be open to doubt. He has expressed his conviction "that it must be possible to find such a direct proof for the compatibility of the arithmetical axioms, by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers," and has formulated the problem as the second in his list of unsolved mathematical problems. See Göttinger Nachrichten, 1900, p. 264; reprinted in the Archiv der Mathematik und Physik, 3d ser., vol. 1 (1901), p. 54; English translation by Dr. Mary W. NEWSON, in the Bulletin of the American Mathematical Society, 2d ser., vol. 8 (1901–02), p. 447.

† See the writer's definition of an Abelian group, p. 27 of the present volume of the Transactions.

‡ The paragraph numbers 11–14 are reserved for use in § 2.

For, take x so that $x \oplus a = c$, where c is an element for which $c \odot c$ is false. Then if $a \odot a$ were true, we should have $x \oplus a \odot x \oplus a$, or $c \odot c$.

From 1, 2, and 18 we have then the important theorem:

19. *Of the three relations:* $a = b$, $a \odot b$, $b \odot a$, *of any two elements* a *and* b, *at least one must be true, and not more than one can be true.*

Definitions.—When $a \odot b$ (a "less than" b), we write also: $b \odot a$ (b "greater than" a). Every element a, not 0, is called *positive* or *negative* according as $a \odot 0$ or $a \odot 0$. Thus (by 17 and 19) all the elements are divided into three classes: the positive elements, the negative elements, and zero.

20. If b is positive, $a \oplus b \odot a$; if b is negative, $a \oplus b \odot a$.

For, by 9 and 15, if $0 \odot b$, then $a \odot a \oplus b$; and if $b \odot 0$, then $a \oplus b \odot a$.

21. If a and b are positive, $a \oplus b$ is positive; if a and b are negative, $a \oplus b$ is negative. (By 20 and 2.)

22. If $a \odot b$, there is a positive x such that $a = b \oplus x$; if $a \odot b$, there is a positive y such that $a \oplus y = b$.

For, take x and y by 8; then both will be positive, since, by 20, if x were negative, we should have $a \odot b$, and if y were negative, $a \odot b$.

23. If the sequence S in 5 is composed of positive elements, the element A (called the upper limit of the sequence) is also positive. (By 2.)

24. If a is positive, there is a positive x such that $x \odot a$. (By 4.)

25. We can now state the important result that *the positive elements form a system of absolute continuous magnitudes with respect to* $\oplus$. For (by 10, 19, and 22) there is at least one positive element; and (by 21, 20, 7, 22, 23 with 6, and 24) the postulates 1–6 in the writer's paper on magnitudes are satisfied.*

26. *Definition.*—By 16 and 17 every element a determines uniquely an element $-a$, such that $a \oplus (-a) = 0$; this element $-a$ is called the *opposite* of a. By 21, if a is positive, $-a$ is negative; if a is negative, $-a$ is positive; and if a is 0, $-a$ is also 0. Any two elements a and b form a pair of opposite elements if $a \oplus b = 0$.

27. From $a \oplus (-c) = b$ follows $a = b \oplus c$, and conversely. (By 7 and 26.)

28. $(-a) \oplus (-b) = -(a \oplus b)$. (By 6 and 7.)

29. If $a \odot b$, then $-a \odot -b$. (By 22, 27, and 20.)

* *Magnitudes*, p. 267.

30. From 26 and 28 we see that *the negative elements will also form a system of absolute continuous magnitudes with respect to* $\oplus$ since the negative elements can be put into one-to-one correspondence with the positive elements, in such a way that if a and b correspond to $-a$ and $-b$, then $a \oplus b$ will correspond to $(-a) \oplus (-b)$.

Sufficiency of the postulates to define an assemblage.

THEOREM II.—*Any two assemblages,* $M(\odot, \oplus)$ *and* $M'(\odot, \oplus)$, *which satisfy the postulates* 1–10 *are equivalent;* that is, they can be brought into one-to-one correspondence in such a way that when a and b in M correspond to a' and b' in M', we shall have:

1°) $a' \odot b'$ whenever $a \odot b$; and

2°) $a \oplus b$ will correspond to $a' \oplus b'$.

Proof.—First make the positive elements of M correspond to the positive elements of M' as in Theorem II of the writer's paper on magnitudes.* Next, make the negative element of M correspond to the negative elements of M', by making a correspond to a' whenever $-a$ corresponds to $-a'$.

Finally, make the zeros of the two systems correspond to each other.

The relation 2°) then clearly holds when a and b are both positive, or both negative, or when either is 0.

To show that it holds also when one is positive and the other negative, let a be positive, b negative, and $a \oplus b = c$; and let a', b', c' be the elements of M' corresponding to a, b, c in M. Then if c is positive, we have $a = c \oplus (-b)$, where a, c, and $-b$ are all positive; whence $a' = c' \oplus (-b')$, or $a' \oplus b' = c'$. If c is negative, we have $b = c \oplus (-a)$, where b, c, and $-a$ are all negative; whence, $a' \oplus b' = c'$, as before. And if $c = 0$, we have $a = -b$, whence $a' = -b'$, or $a' \oplus b' = 0 = c'$.

Thus relation 2°) is always satisfied.

To prove relation 1°), notice that if $a \odot b$ there is a positive x such that $a \oplus x = b$. Then $a' \oplus x' = b'$, where x' is also positive. Hence $a' \odot b'$.

From Theorems I *and* II *we see that the postulates* 1–10 *define essentially a single assemblage,* which we may call *the system of real quantities with respect to* $\odot$ *and* $\oplus$. The relation $\odot$ may then be called the relation of (algebraically) *less than*, and the rule of combination $\oplus$, the *addition of real quantities.*

For example, each of the systems mentioned in the proof of Theorem I is a system of real quantities in the sense here defined.

* *Magnitudes*, p. 277.

Independence of the postulates.

THEOREM III.—*The postulates* 1–10 *are independent;* that is, no one is a consequence of the other eight. This is shown by the following systems, each of which satisfies all the other postulates, but not the one for which it is numbered.*

(1) The system of all real numbers, with $\odot$ so defined that $a \odot b$ is true when and only when a and b are positive or zero, and $a < b$; and $a \oplus b$ defined as $a + b$.

(2) The system of all real numbers, with $a \odot b$ defined as $a \neq b$, and $a \oplus b$ as $a + b$.

Here 2 is not satisfied, for from $a \odot b$ and $b \odot a$ it does not follow that $a \odot a$. In 5, take $A = c$.

(3) The system of all real numbers, with $a \odot b$ defined as $a \leqq b$, and $a \oplus b$ as $a + b$.

(4) The system of all integral numbers, with $\odot$ defined as $<$, and $\oplus$ as $+$.

Here 5 is satisfied vacuously (to use a term of Professor MOORE's), since no infinite sequence S of the kind described can occur.

(5) The system of all rational numbers, with $\odot$ defined as $<$, and $\oplus$ as $+$.

To show that 5 fails, consider any infinite sequence of increasing rational fractions used to define $\sqrt{2}$.

(6) The system of all positive real numbers, with $\odot$ defined as $<$, and $\oplus$ defined by the relation $a \oplus b = b$.

(7) The system of all positive real numbers, with $\odot$ defined as $<$, and $\oplus$ defined by the relation $a \oplus b = \sqrt{ab}$.

Here 7 fails, since $(2 \oplus 2) \oplus 8 = 4$, while $2 \oplus (2 \oplus 8) = \sqrt{8}$. In 8, take $x = b^2/a$.

(8) The system of all positive real numbers, with $\odot$ defined as $<$, and $\oplus$ as $+$.

(9) The system of all real numbers, with $\odot$ so defined that $a \odot b$ is always true when $b \neq 0$ and false when $b = 0$; and $a \oplus b$ defined as $a + b$.

Here 2 holds, for since $b \odot c$, c cannot be 0. In 3, take $c = 0$. In 4 and 5, x and A may be any elements not 0. To show that 9 fails, take $y = -a$.

*On this method, used by PEANO and HILBERT, see especially PADOA, *loc. cit.*

(10) A system comprising only one element a, with $\circledcirc$ and $\oplus$ so defined that $a \circledcirc a$ is false, and $a \oplus a = a$.

Remarks on the further development of the theory.

In the present paper we have drawn only such deductions from the postulates 1–10 as were needed to establish Theorem II. The first step in the further development of the theory would be naturally the definition of multiplication. Any positive quantity being chosen as a "unit," and denoted by 1, every positive element can be expressed as the limit of an infinite sequence of rational fractions of the form $m1/n$, where m and n are positive integers.*

The product $a \cdot b$, or ab, of any two real quantities a and b is then defined as follows:

1°) when a and b are positive,

$$ab = \lim \left(\frac{mp}{nq} 1 \right),$$

where $\lim (m1/n) = a$ and $\lim (p1/q) = b$;

2°) when a or b is negative, $ab = -[(-a)b] = -[a(-b)]$;

3°) when a or b is zero, $a0 = 0b = 0$.

The commutative, associative and distributive laws for the multiplication of real quantities then follow readily from the corresponding properties of positive integers; and the quotient b/a, where $a \neq 0$, is then uniquely defined as that quantity x which satisfies the relation $ax = b$.

§2. The second set of postulates for real quantities.

We consider in the second place an assemblage in which $\circledcirc$, $\ominus$, and $\odot$ are so defined that the following fourteen postulates are satisfied:

1–10. *The same as postulates* 1–10 *in* §1.

11. *If a and b belong to the assemblage, then $a \odot b$ does also.*

12. *First distributive law: $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$, whenever a, b, c, $a \odot b$, etc., all belong to the assemblage.*

13. *Second distributive law: $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$, whenever a, b, c, $a \oplus b$, etc., all belong to the assemblage.*

14. *If 0 is an element such that $c \oplus 0 = c$ for every element c, and if $0 \circledcirc a$ and $0 \circledcirc b$, then $0 \circledcirc a \odot b$, whenever $a \odot b$ belongs to the assemblage.*

Consistency of the postulates of the second set.

Theorem I′.—*The postulates* 1–14 *are consistent with one another.*

For, one assemblage which satisfies them all is the system of all real numbers, with $\circledcirc = <$, $\oplus = +$, and $\odot = \times$. Another such assemblage is the system

*Cf. 39, below.

of all positive real numbers, with $\circledcirc = <$, $\oplus = \times$, and $\odot$ defined as follows: $a \odot b = 2^{\alpha\beta}$, where α and β are real numbers such that $a = 2^{\alpha}$ and $b = 2^{\beta}$.

Deductions from the postulates of the second set.

As consequences of the postulates 1–14 we have in the first place all the propositions 15–30 in § 1, and further, the propositions 31–44, below. The object of the work is to establish the commutative and associative laws for the operation $\odot$, and thus prepare the way for Theorem II′.

31. If a and b are positive, $a \odot b$ will be positive. (By 14, with 11, 17, 19.)

32. For every element a, $a \odot 0 = 0 = 0 \odot a$.

For, $a \odot b = a \odot (b \oplus 0) = (a \odot b) \oplus (a \odot 0)$, by 11 and 12; hence $a \odot 0 = 0$, by 17. Similarly, $0 \odot a = 0$, using 13 in place of 12.

33. $a \odot (-b) = -(a \odot b)$ and $(-a) \odot b = -(a \odot b)$.

For $(a \odot -b) \oplus (a \odot b) = a \odot (-b \oplus b) = a \odot 0 = 0$, by 12, 26, and 32. Hence the first part of the theorem is true by 26. The second part follows in a similar way, if 13 is used in place of 12.

34. $(-a) \odot (-b) = a \odot b$. (By 33 and 26.)

35. If a and b are both positive, or both negative, $a \odot b$ is positive; if one is positive and the other negative, $a \odot b$ is negative. (By 31, 34, and 33.) Hence, if $a \odot b = 0$, then either $a = 0$ or $b = 0$.

36. If c is positive, and $x \circledcirc y$, then $c \odot x \circledcirc c \odot y$, and $x \odot c \circledcirc y \odot c$.

For, by 22 take a positive z so that $x \oplus z = y$; then $c \odot z$ is positive, by 31. Now $(c \odot x) \oplus (c \odot z) = c \odot (x \oplus z) = c \odot y$, by 12. Hence $c \odot x \circledcirc c \odot y$, by 20. The second part of the theorem follows in a similar way, if 13 is used in place of 12.

37. If a and b are positive, and $a \circledcirc a'$, $b \circledcirc b'$, then $a \odot b \circledcirc a' \odot b'$. (By 36.)

Since (by 25) the positive elements form a system of absolute continuous magnitudes, we may use the theorems concerning multiples and sub-multiples for such elements.* Hence:

* *Magnitudes*, 13 34.

38. If a and b are positive,

$$\left(\frac{m}{n}a\right)\odot\left(\frac{p}{q}b\right)=\frac{mp}{nq}(a\odot b).$$

For, by mathematical induction from 12 and 13 we have $a\odot(mb)=m(a\odot b)$ and $(ma)\odot b=m(a\odot b)$, whence, replacing b by b/m in the first equation, and a by a/m in the second, we obtain:

$$a\odot(b/m)=(a\odot b)/m \qquad \text{and} \qquad (a/m)\odot b=(a\odot b)/m.$$

From these four equations the theorem is readily inferred.

39. If c is any fixed positive element, any positive element a may be expressed in the form: *

$$a=\lim_{n=\infty}\left(\frac{m}{n}c\right),$$

where the positive integer m corresponding to any positive integer n is determined by the following relation:

$$\frac{m}{n}c \leqq a \leqq \frac{m+1}{n}c.$$

40. If a, b, c are positive, and

$$a=\lim\left(\frac{p}{n}c\right) \qquad \text{and} \qquad b=\lim\left(\frac{q}{n}c\right),$$

then

$$a\odot b=\lim\frac{pq}{nn}(c\odot c).$$

For, by 37 and 39 we have:

$$\left(\frac{p}{n}c\right)\odot\left(\frac{q}{n}c\right)\leqq a\odot b\leqq\left(\frac{p+1}{n}c\right)\odot\left(\frac{q+1}{n}c\right),$$

whence, by 38,

$$\frac{pq}{nn}(c\odot c)\leqq a\odot b\leqq\frac{pq}{nn}(c\odot c)\oplus\frac{1}{n}\left(\frac{p+q+1}{n}\right)(c\odot c),$$

in which the last term can be made as small as we please by increasing n, since p/n and q/n remain finite.

* A detailed elementary discussion of this familiar application of the theory of limits is given in the writer's dissertation, *Ueber die Grund-Operationen an absoluten und complexen Grössen*, Braunschweig (1901), chapters II and III.

In view of 40, the commutative and associative laws for $\odot$ follow at once from the corresponding properties of positive integers — first when all the elements are positive, and then, by 32, 33, and 34, when any of the elements concerned are negative or zero. Hence, in all cases:

41. $a \odot b = b \odot a$, and:

42. $(a \odot b) \odot c = a \odot (b \odot c)$.

From 38 we derive also the following theorem:

43. For every two elements a and b, provided $a \neq 0$, there is an element x such that $a \odot x = b$. $\quad (x = b/a.)$

Proof.—If a and b are positive, and c is any fixed positive element, there is, by 38, a positive integer n_1 such that $a \odot (c/n) \oslash b$ for every positive integer $n > n_1$; and for every such n there is a positive integer m such that

$$a \odot \frac{m}{n} c \oslash b \oslash a \odot \frac{m+1}{n} c.$$

The sequence $(m/n)c$ will then have a limit, x, which will be the required element.

If a [or b] is negative, take $x = -z$, where z is the positive element such that $(-a) \odot z = b$ [or $a \odot z = (-b)$]. If $b = 0$, take $x = 0$.

In view of 41, 42, and 43 we see that *the elements of our assemblage, excluding the element* 0, *form an Abelian group with respect to* $\odot$.* Hence (using also 32):

44. There is one and only one unit-element, 1, such that $a \odot 1 = a$ for every element a. In particular, $1 \odot 1 = 1$.

We notice also, in passing, that every assemblage which satisfies postulates 1–14 will form a *field*† *with respect to* $\oplus$ *and* $\odot$.

Sufficiency of the postulates to define an assemblage.

THEOREM II′.—*Any two assemblages,* $M(\oslash, \oplus, \odot)$ *and* $M'(\oslash, \oplus, \odot)$, *which satisfy the postulates* 1–14 *are equivalent;* that is, they can be brought into one-to-one correspondence in such a way that when a and b in M correspond to a' and b' in M' we shall have:

* Cf. 15–17, above.

† See the definitions of a field by L. E. DICKSON and by E. V. HUNTINGTON, p. 13 and p. 31 of the present volume of the Transactions.

1°) $a' \odot b'$ whenever $a \odot b$;

2°) $a \oplus b$ will correspond to $a' \oplus b'$; and

3°) $a \odot b$ will correspond to $a' \odot b'$.

Proof.—By § 1, Theorem II, place M and M' in one-to-one correspondence in such a way that relations 1°) and 2°) are satisfied, the elements 0, 1 in M being made to correspond to the elements $0'$, $1'$ in M'. Then 3°) will also be satisfied.

For, consider first any two positive elements, a and b, in M, and the two corresponding positive elements, a' and b', in M'; and let $a = \lim (p/n)\,1$ and $b = \lim (q/n)1$. Then $a' = \lim (p/n)1'$ and $b' = \lim (q/n)1'$. Hence, by 40 and 44, $a \odot b = \lim (pq/nn)1$ and $a' \odot b' = \lim (pq/nn)1'$, which are corresponding elements.

When either a or b is negative or zero, 3°) is then clearly satisfied in view of 32, 33, and 34.

From Theorems I′ *and* II′ *we see that the postulates* 1–14 *define essentially a single assemblage*, which we may call *the system of real quantities with respect to* $\odot$, $\oplus$, *and* $\odot$. The relation $\odot$ may then be called *less than*, and the rules of combination $\oplus$ and $\odot$, *addition* and *multiplication* of real quantities.

The definition of the system of real quantities in § 1 is clearly consistent with that in § 2; in the first case multiplication is treated as a derived concept, in the second, as a fundamental concept.

Independence of the postulates of the second set.

THEOREM III′.—*The postulates* 1–14 *are independent.*

The systems [1]–[9] which we use to prove the independence of the first nine of these fourteen postulates are the same as the systems (1)–(9) above, with $\odot$ defined by the relation $a \odot b = ab$.

Here it should be especially noticed that the systems [6] and [7] satisfy postulates 12, 13, and 14 (the last vacuously); and that the systems [2], [3], and [9] satisfy postulate 14.

To prove the independence of postulate 10 we use a system [10] comprising only one element a, with $\odot$, $\oplus$, and $\odot$ so defined that $a \odot a$ is false, $a \oplus a = a$, and $a \odot a = a$.

To prove the independence of postulates 11–14, we use the system of all real numbers, with $\odot = <$, $\oplus = +$, and $\odot$ defined as follows:

[11] $a \odot b = ab$ when a and b are rational; $a \odot b$ not in the assemblage when either a or b is irrational.

[12] $a \odot b = ab$ when $b \neq 0$; and $a \odot 0 = a$.

Here 12 fails, since we have, for example, $2 \odot (3 \oplus -3) = 2 \odot 0 = 2$, while $(2 \odot 3) \oplus (2 \odot -3) = 6 \oplus (-6) = 0$. To show that 13 is satisfied, consider the two cases: $c = 0$ and $c \neq 0$.

[13] $a \odot b = ab$ when $a \neq 0$; and $0 \odot b = b$.

Here 13 fails, since we have, for example, $(3 \oplus -3) \odot 2 = 0 \odot 2 = 2$, while $(3 \odot 2) \oplus (-3 \odot 2) = 6 \oplus -6 = 0$. To show that 12 is satisfied, consider the two cases $a = 0$ and $a \neq 0$.

[14] $a \odot b = -ab$.

Here 12 and 13 are clearly satisfied, while 14 is not.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.